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ਪਟਿਆਲਾ

The Motto of Our University

(SEWA)

SKILL ENHANCEMENT

EMPLOYABILITY

WISDOM

ACCESSIBILITY

SELF-INSTRUCTIONAL STUDY MATERIAL FOR JGND PSOU

JAGAT GURU NANAK DEV
PUNJAB STATE OPEN UNIVERSITY, PATIALA

(Established by Act No. 19 of 2019 of the Legislature of State of Punjab)

BACHELOR OF ARTS

DISCIPLINE SPECIFIC ELECTIVE (DSE) ECONOMICS

SEMESTER – VI

(BAB33601T) QUANTITATIVE METHODS

Head Quarter: C/28, The Lower Mall, Patiala-147001
WEBSITE: www.psou.ac.in

The Study Material has been prepared exclusively under the guidance of Jagat Guru Nanak Dev Punjab State Open University, Patiala, as per the syllabi prepared by Committee of Experts and approved by the Academic Council.

JAGAT GURU NANAK DEV PUNJAB STATE OPEN UNIVERSITY, PATIALA
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PREFACE

Jagat Guru Nanak Dev Punjab State Open University, Patiala was established in December 2019 by Act 19 of the Legislature of State of Punjab. It is the first and only Open Universityof the State, entrusted with the responsibility of making higher education accessible to all, especially to those sections of society who do not have the means, time or opportunity to pursue regulareducation.

In keeping with the nature of an Open University, this University provides a flexible education system to suit every need. The time given to complete a programme is double the duration of a regular mode programme. Well-designed study material has been prepared in consultation with experts in their respective fields.

The University offers programmes which have been designed to provide relevant, skill-based and employability-enhancing education. The study material provided in this booklet is self-instructional, with self-assessment exercises, and recommendations for further readings. Thesyllabus has been divided in sections, and provided as units for simplification.

The University has a network of 110 Learner Support Centres /Study Centres, to enable students to make use of reading facilities, and for curriculum-based counseling and practicals. We, at the University, welcome you to be a part of this institution of knowledge.

Dean Academic Affairs

BACHELOR OF ARTS
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MAX. MARKS:100

EXTERNAL:70

INTERNAL:30

PASS:40%

TOTAL CREDITS:6

Objective:

This course introduces the students with the applications of mathematical techniques to economic theory. It also tells about the statistical methods to analyse the data.

INSTRUCTIONS FOR THE PAPER SETTER/EXAMINER:

1. The syllabus prescribed should be strictly adhered to.
2. The question paper will consist of three sections: A, B, and C. Sections A and B will have four questions each from the respective sections of the syllabus and will carry 10 marks each. The candidates will attempt two questions from each section.
3. Section C will have fifteen short answer questions covering the entire syllabus. Each question will carry 3 marks. Candidates will attempt any 10 questions from this section.
4. The examiner shall give a clear instruction to the candidates to attempt questions only at one place and only once. Second or subsequent attempts, unless the earlier ones have been crossed out, shall not be evaluated.
5. The duration of each paper will be three hours.

INSTRUCTIONS FOR THE CANDIDATES:

Candidates are required to attempt any two questions each from the sections A, and B of the question paper, and any ten short answer questions from Section C. They have to attempt questions only at one place and only once. Second or subsequent attempts, unless the earlier ones have been crossed out, shall not be evaluated.

Section - A

Unit 1: Differentiation of Functions: Simple and Partial Derivatives, Differentiation of

Simple functions – Polynomial (x) and Exponential functions. Maxima and Minima of functions of one variable only. Their Applications of Micro and Macro Economics.

Unit 2: Matrices: Definition and Types, Operations (Sum, difference) Product and Transpose.

Unit 3: Adjoint and inverse of a matrix (upto 3x3) Solution of simultaneous equations (up to 3) by matrix methods and Crammer's Rule.

Unit 4: Data and Methods: Types of data; Method of Data Collection.

Section – B

Unit 5: Measures of Central Tendency: Mean, Median, Mode

Unit 6: Measures of Dispersion and Skewness.

Unit 7: Correlation Analysis: Karl Pearson's (excluding grouped data) and Spearman's rank formula.

Unit 8: Simple Regression Analysis: regression meaning, properties, X on Y and Yon X

Suggested Readings:

1. K. Sydsaeter and P. Hammond (2002). Mathematics for Economic Analysis, Pearson Educational Asia: Delhi.
2. Wainwright, Chiang, Fundamental Methods of Mathematical Economics, Tata McGraw Hill, 2013.
3. Archibald, C.C and Lipsey, R.G: An Introduction to a Mathematical Treatment of Economics, 1977, English Language Book Society.
4. Sanchati, D.C. and Kapoor, V.K: Business Mathematics, New Delhi, Sultan Chand & Sons, 1993.
5. Gupta, S.C.: Fundamentals of Statistics, Bombay, Himalaya Publishing House.

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BACHELOR OF ARTS

DISCIPLINE SPECIFIC ELECTIVE (DSE) ECONOMICS

SEMESTER – VI

(BAB33601T) QUANTITATIVE METHODS

SECTION A

UNIT No.:	UNIT NAME
UNIT 1	Differentiation of Functions
UNIT 2	Matrices
UNIT 3	Adjoint and inverse of a matrix
UNIT 4	Data and Methods

SECTION B

UNIT No.:	UNIT NAME
UNIT 5	Measures of Central Tendency
UNIT 6	Measures of Dispersion and Skewness.
UNIT 7	Correlation Analysis
UNIT 8	Simple Regression Analysis

BACHELOR OF ARTS

SEMESTER – VI

QUANTITATIVE METHODS

UNIT 1: DIFFERENTIATION OF FUNCTIONS

STRUCTURE

1.0 OBJECTIVES

After studying the Unit, students will be able to:

- Meaning of determination
- Various function in economics
- Methods to solve equations
- Limits of function and sequence
- Continuity of function and its types
- Types of differentiation

1.1 FUNCTIONS IN ECONOMICS

Knowledge of functions plays an important role in economics. It shows the relationship between dependent (output) variables and dependent (input) variables and is helpful to know how to obtain values for dependent variables from independent variables. The dependent variables are called effects, and the independent variables are called causes.

Generally, an independent variable is plotted on the horizontal axis, and a dependent variable is plotted on the y-axis. If one value of an independent variable maps two or more values of dependent variables, then it can't be considered a function as per the vertical line test method. These functions in set algebra can be classified as injective (one-one), many-one, surjective (onto), algebraic, into, periodic, integer, signum, polynomial, linear, identical, quadratic, cubic, composite, modulus, fractional, constant, and even and odd functions.

Functions are generally represented as follows:

$$Y=f(X)=2x^2$$

Where f of X is equal to the double of X^2. Here, we can input values for an independent variable X to obtain values for the dependent variable Y.

The algebraic operations performed to solve equations follow the BEDMAS principle. It means first completing inside bracket operations, followed by exponentiation, multiplication, division, addition, and subtraction. These are explained as follows:

Exercise 1: Let Y=f(X) =5+3X^{2,} Find value of f (2).

Solution: Here, we first substitute the value of X = 2 and square it, giving 4. Now we can multiply this 4 by 3 obtaining 12. Finally, we add this result to 5 giving value of 17. Hence, the value of Y at X = 2 is 17.

Exercise 2: Let Y = f(X) = (5+3) X^{2,} Find value of f (3).

Solution: In the above example, we need to do the bracket first. Add 5 and 3, giving a value of 8. Later, operate exponentially (X^2), put the value of X = 3, giving 9. Now multiply 8 by 9 giving 72 answers. Here, we need to be careful that X^2=X.X and not 2X. The reciprocal of X means 1 divided by X and exponential means exponential power time multiplication of the value.

The values of variables can also be substituted to find the final value of any economic indicator. For example:

Exercise 3: Let, Import=X and Export=Z. The value of Import (X) is constant= Rs. 300 million and Exports in Rs. million is (Z)= 500+0.2Y. Calculate net income.

Solution: The net income can be calculated by finding the difference between exports and imports.

Net Income= Exports-Imports

Net Income=(500+0.2Y)-300

=200+0.2Y

We need to note that multiplication of a negative value with a positive value or multiplication of a positive value with a negative value will always give a negative result. If a negative value is multiplied even number of times, it will yield a positive value, and if a negative value is multiplied by odd number of times, it will yield a negative value. Also, if any value is multiplied with zero will yield a zero value, and if zero is divided by any value, it will also yield zero. However, any value divided by zero will yield an infinite value, which is not defined.

For example, total and average revenue can be calculated based on the quantity of goods sold at a given price. Let a firm sell Q quantities of goods for P. Total revenue is calculated as follows:

$$\text{Total revenue (TR)} = \text{Price (P). Quantity (Q) sold} = P.Q$$

$$\text{The total Profit} = \text{TR-Total Cost (C)} = P.Q-C$$

$$\text{Average Profit or Per unit Profit} = (P.Q-C)/Q$$

Students of economics need to deal with various types of functions. The cost function is a major function that gives the total cost of producing a Q number of the same goods. Generally, it consists of two costs components called fixed costs and variable costs. It is calculated as follows:

$$\text{Total Cost (C)} = \text{Fixed Cost} + \text{Variable Cost}$$

Exercise 4: A company produces cold drinks. The building and infrastructure costs are fixed. The raw material, energy and labour costs are considered as variable costs. If the fixed cost is Rs. 20,000 and variable cost is Rs. 2 per bottle of soft drink. Calculate total cost.

Solution: The total cost is calculated as follows:

$$\text{Total Cost (C)} = \text{Fixed Cost} + \text{Variable Cost}$$

$$\text{Total Cost (C)} = 20{,}000 + 2 \text{ times number of bottles produced (Q)}$$

$$C = 20{,}000 + 2Q$$

The variable cost associated with the production of one additional unit is called the marginal cost. The marginal revenue is the slope of linear revenue function, and the marginal profit is the slope of the profit function. Companies calculate the break-even point to make business decisions, where at the break-even point, profit is zero and cost is equal to revenue.

The major functions in the field of economics include constant functions, polynomials, and rationals; these are discussed in the next section.

1.1.1 Constant Function

The constant functions assume only one value. The output value is independent of the input value. The graphs of constant functions are as follows:

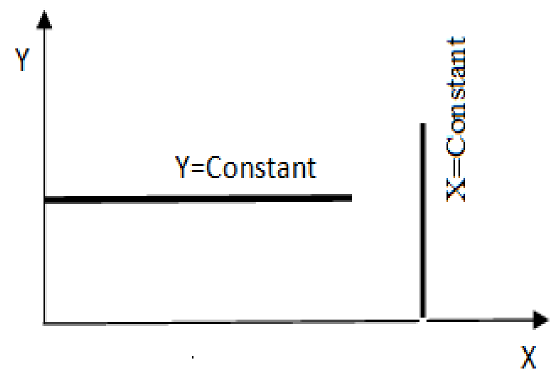


Figure 1: Constant functions

For example, the function Y = f(X) = 3 or Y = 3 shows that the domain of this function is set of all real numbers, but the co-domain is fixed equal at 3. Means f (2) = 3, f(5) = 3, and for any value of X, Y assumes only one value fixed to 3. In the case of one variable polynomial X, the non-zero constant assumes a polynomial of zero degree, and its line is parallel to the X-axis and never touches it. However, if f(X) = 0, is the identically zero function with line resting on the X-axis. Constant functions or even functions are symmetric with the Y-axis or have zero slope, as shown in the figure above. The derivative of such functions is zero. The real example is a store where every item costs a fixed price. The graph of such functions assumes a linear line parallel to the X-axis.

1.1.2 Polynomial Function

A polynomial function assumes positive powers only in the functional relationships. Generally, a polynomial is expressed as:

$$Y=f(X)=C_0+C_1X+C_2X^2+C_3X^3+C_4X^4+\dots$$

Where C_0, C_1, C_2, etc. are constants and X and Y are independent and dependent variables. It will be a constant function if the power of X is zero and a linear function if the power of X is 1. If the power of X is 2, it is a quadratic, and if the power of X is 3, it is a cubic function.

1.1.3 Quadratic Functions

The word quadratic means square. It has a parabola graph and is useful to identify maxima and minima in economic optimisation problems by twice differentiating functions. These are useful in profit and loss, forecasting, trajectory plots, etc. The quadratic functions are expressed as:

$f(X) = aX^2 + bX + c$, where a, b, and c are constants not equal to zero. The value of X can be calculated for equation the $aX^2 + bX + c=0$ using the following formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

1.1.4 Cubic Functions

The cubic function was discovered by Scipione del Ferro, an Italian mathematician. In this function, each value of X has one unique Y-value, like a one-to-one function. It has no slope and assumes an S-shape. The change in volume of a ball or cube is a good example of cubic function. The cubic equation is indicated by the following:

$$\mathbf{f(X) = a\,X^3 + b\,X^2 + c\,X + d}.$$

1.1.5 Rational Functions

A rational function is the ratio of two polynomial functions in the form $\frac{P}{Q}$ where Q is not equal to zero. Mathematically, a rational function can be represented as $f(x) = \frac{P(x)}{Q(x)}$ where $Q(x) \neq 0$. For example, $f(x)= \frac{x^2+x-1}{2x^2+2x-5}$ where $2x^2 - 2x - 5 \neq 0$, is a rational function. An asymptote of a curve is a line that shows the distance between the curve and the line approaches zero as one of the coordinates along the x-axis or y-axis approaches infinity. Rational function curves can have three types of asymptotes: horizontal asymptote, for example X=a; vertical asymptote, for example Y=b; and oblique asymptote for example Y=ax+b. The most common example of a rational function in economics is the average cost. Where, Average Cost= $\frac{C(x)}{x}$ Here, C(x) is the total cost of x number of items.

Exercise 5: A chocolate manufacturing plant has a fixed cost of ₹5000 and per unit variable cost of ₹25. Find the average cost when 200 chocolates are produced.

Solution: Here the total cost function C(x)= 5000+25x and Average Cost= $\frac{5000+25x}{x}$

Average Cost= $\frac{5000+25(200)}{200} = 50$ per item

1.1.6 Non-algebraic Functions

Non-algebraic functions are also called transcendental functions. These transcendental functions can be expressed in algebra in the form of infinite sequences. Non-algebraic functions are polynomials in the form of trigonometric, for example, f(x)=Sin (2x+3); absolute value functions, for example, f(x)=l5l; logarithmic functions, for example, f(x)=log (x); and exponential functions, for example, $f(x)= 4x^3$; and powered as a root, for example, $f(x) = \sqrt{x}$. In these functions, the behavior of the left-hand side variable is explained by the variable(s) on the right-hand side. Knowledge of these functions will help in understanding business functional relationships for short-term and long-term decision-making.

1.2 SOLVING EQUATIONS

A function is used to define the relationship among dependent and independent variables. The solution of a functional relationship will be obtained with the help of an equation. An equation is a mathematical relation that shows the equality of two sides of an expression. For instance, 4X+3Y=15. The left and right sides are equal. The rate of change in the dependent variable with respect to the independent variable shall be calculated by taking the derivative of the dependent variables with respect to one of the independent variables. An equation that involves independent and dependent variables and has at least one derivate of the dependent variable with respect to the independent variable is called a differential equation. There are many methods to solve equations. The main methods to solve equations include algebraic, graphical, substitution, elimination, and matrix methods. Here, we can add, subtract, divide and multiply both sides of an equation to simplify and solve it.

1.2.1 Algebraic Method

Consider the following example:

$$5x + 20 = 6x + 9$$

It has only one variable X. Here we can subtract 6X from both sides. This gives us the following:

$$5X + 20 - 6X = 6X + 9 - 6X$$

$$-X+20=9$$

Now we can subtract 20 from both sides yielding

$$- X + 20 - 20 = 9 - 20$$

$$- X = - 11$$

Multiply both sides by -1 we get

$$- 1 (-X) = - 1 (-11)$$

$$X = 11$$

To test the value put X=11 in the equation 5X + 20 = 6X + 9 and we have 5 (11) + 20 = 6 (11) + 9 = 75. Now, both sides have value equal to 75 and solution value X = 11 is valid.

1.2.2 Graphical Method

The Graphical method is helpful in finding solutions to equations where two variables are involved because a graph paper has only two dimensions, i.e., length and breadth. This method helps in investigating economic phenomena using diagrams and graphs. The abstract graphical relations help strategists manipulate and gain better insights into the business environment.

For example:

$$Y=2X-1 \text{ and } 2Y= -X+8$$

Here, we can solve this function by equating these two equations and simplifying as follows:

$$2X-Y-1=0 \quad \text{---Equation (1)}$$

$$X+2Y-8=0 \quad \text{---Equation (2)}$$

$$Y=1-2X \quad \text{--from equation (1)}$$

$$Y = \frac{8-X}{2} \quad \text{---from equation (2)}$$

The graphical behaviour of variable Y can be plotted for equations (1) and (2). As it is a linear function with the exponential power of variables in both equations equal to 1. We need to plot at least two points to draw a line. Here, we can assume Y = 0 to get one point for X variable and assume X = 0 to get another point for Y the variable.

The X and Y values for equation (1) are as follows:

Variable	X (when Y=0)	Y (When X=0)
Y	0	1
X	1/2	0

Similarly, the X and Y values for equation (2) are as follows:

Variable	X (when Y=0)	Y (When X=0)
Y	0	4
X	8	0

Now we can plot the graph as shown below:

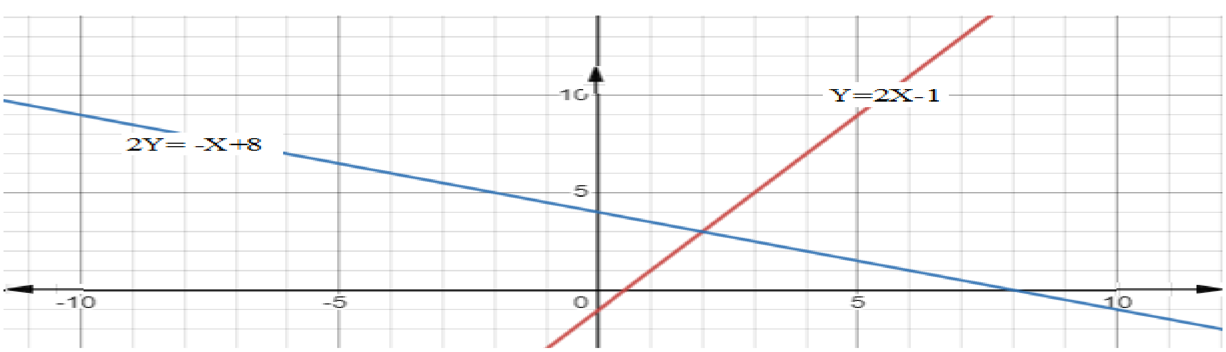


Figure 2: Graph of functions Y=2X-1 and 2Y= -X+8

1.2.3 Elimination by Substitution Method

This method involves eliminating one variable from another equation by substituting its value in the equation. For example, we have two linear equations:

$$3x+2y = 11 \text{ and } 2x-y = 4$$

We have

$$3x+2y = 11 \qquad \ldots (1)$$

$$2x-y = 4 \qquad \ldots (2)$$

We can modify the equation (2) as follow:

$$x = (y+4)/2=y/2 + 2 \ldots (3)$$

The value of x obtained in equation (3) will be substituted in equation (1) as follow:

$$3 (Y/2 + 2) +2y = 11$$

$$3Y/2 + 6 + 2Y=11$$

$$Y (3/2 + 2)=11-6$$

$$7Y/2=5$$

$$Y=5*2/7= 10/7$$

Now, using the distributive property of equations, we can substitute this value of Y in either equation (1) or equation (2). Let us substitute this value in equation (2), and we have the following:

$$2x - y = 4$$

$$2x - 10/7 = 4$$

$$2x = 4 + 10/7 = 38/7$$

$$x = 38/ (2*7) = 38/14 = 19/7$$

Hence, $x = 19/7$ and $y = 10/7$ satisfy both the equations (1 and 2).

1.2.4 Matrix Method

The graphical method is limited to solving equations involving a maximum of two variables. The elimination method is quite a lengthy exercise to find the values of variables satisfying all equations. The matrix method has an advantage over these two methods as it is quick and easy to find solutions to equations involving two or more variables. This method is based on the Gaussian elimination method to find solutions. The following are important steps to solve equations using the matrix method:

1. Write variables in equations in the proper order.
2. The coefficients and constructs should be written on their respective sides.

3. It should be possible to find the inverse of the matrix, i.e., determinant of the matrix should not be zero.

The following example will explain this method:

Let there be three equations,

$$a_1x+a_2\,y+a_3z=k_1$$

$$b_1x+b_2\,y+b_3z=k_2$$

$$c_1x+c_2\,y+c_3z=k_3$$

Where a, b, c, and k are constants and x, y, and z are variables.

$$\begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} \begin{matrix} = \\ = \\ = \end{matrix} \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

$$AX=K$$

Where Matrix A presents constants, matrix X presents variables and matrix K presents resource constraints.

$$A = \begin{pmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{pmatrix}$$

$$X = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$K = \begin{pmatrix} k_1 \\ k_2 \\ k_3 \end{pmatrix}$$

$$X=A^{-1}K$$

Where $A^{-1} \neq 0$ and A^{-1} =Adjoint A/Determinant of A

For example, let us have,

$$2x+y+2z=2$$

$$2x-y+z=12$$

$$x+3y-z=6$$

$$A = \begin{pmatrix} 2 & 1 & 2 \\ 2 & -1 & 1 \\ 1 & 3 & -1 \end{pmatrix} \qquad X = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad K= \begin{pmatrix} 2 \\ 12 \\ 6 \end{pmatrix}$$

A^{-1} =Adjoint A/Determinant of A

$|A|$ = Determinant of A= 2(-1*-1 - 3*1) - 2(1*-1 - 2*3) + 1(1*1 - (-1) *2)

=2(1-3) - 2(-1-6) + 1(1+2) = -4+14+3=13

The value of the determinant of A is not equal to zero, hence, we can find its inverse. We need to find determinant of the 2X2 matrix, which is as follows:

Row 1 elements are represented by 'a' with subscripts. The first subscript shows the row and the second subscript shows the column as follows:

$a_{11=}$ (-1*-1-3*1) = 1-3=-2; $a_{12=}$ (2*-1-1*1) =-2-1=-3 a_{13}= (2*3-(-1) *1) =6+1=7

Similarly for Row 2 and Row 3, co-factors are as follows:

$a_{21=}$ (1*-1-3*2) = -1-6=-7; $a_{22=}$ (-1*2 -1*2) =-2-2=-4 a_{23}= (2*3-1*1) =6-1=5

$a_{31=}$ (1*1-2*-1) = 1+2=3; $a_{32=}$ (1*2 -2*2) =2-4=-2 a_{33}= (2*-1-2*1) =-2-2=-4

The matrix of these determinants with cofactors shall be calculated by summing the row number and column number of the respective element. If the sum is even, we need to multiply the values of determinants obtained by the + sign, and if the sum of row plus column position comes to an odd number, we need to multiply by the - sign. Now the cofactor matrix obtained is as follows:

Cofactor matrix of A= $\begin{pmatrix} -2 & 3 & 7 \\ 7 & -4 & -5 \\ 3 & 2 & -4 \end{pmatrix}$

The adjoint of A is formed by changing the first row into the first column, the second row into the second column, and the third row into the third column as follow:

Adjoint of A= $\begin{pmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ -7 & -5 & -4 \end{pmatrix}$

A^{-1} =Adjoint A/Determinant of A=1/13* $\begin{pmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ 7 & -5 & -4 \end{pmatrix}$

Now X = A^{-1} K. Here, we need to multiply 3by3 matrix with column matrix K the multiplication result is as follows:

$$\begin{pmatrix} x \\ y \\ \end{pmatrix} = 1/13 \begin{pmatrix} -2 & 7 & 3 \\ 3 & -4 & 2 \\ & & \end{pmatrix} \begin{pmatrix} 2 \\ 12 \\ \end{pmatrix} = \begin{pmatrix} 98/13 \\ -30/13 \\ \end{pmatrix}$$

z	7	-5	-4	6	-70/13

Now, putting these values in all three equations in these questions makes the left side equal to the right side.

1.3 LIMIT OF A FUNCTION

The idea of a limit is the basis of calculus, and a limit is the function value as that function approaches the specific value of a decision variable. The function can be continuous or discontinuous. A function is continuous if the right-side limit and left-side limit values of a function at a specific point x = k exist and are equal. Otherwise, it is discontinuous. Let us take a function f(x)=x-2/x-2; here the function is not defined at x=2 because it will give a value of zero divided by zero, which is not defined. Otherwise, for any value of x, the function f(x) approaches 1. This is the limit; as x approaches 2 but is not equal to 2, the value of the function is 1. In another example, let us have the following function:

$$S(x) = \begin{cases} x^2 & x \neq 2 \\ 1 & x = 2 \end{cases}$$

This function is discontinuous at x = 2 as the given value of a function at x = 2 is 1. However, as $x \rightarrow 2$ the value of the function approaches 4. This can help us to understand the limits and continuity. This function is discontinuous at x = 2. A limit is represented as shown below:

$$\lim_{x\to c} = f(x) = k$$

In the above function, the value of f(x) approaches k as x approaches c. Limit is the output value of a function or sequence is an important part of integration and differentiation to graph functions to understand its real time behaviour. A limit could be one-sided, which means that a function or sequence gives output value as the input approaches a particular value either from the right-side (above) or the left side (below). These left and right-side limits are written as follows:

$$\text{Lim}_{x\text{-->}c^+} f(x) = k \text{ or } f(x) \rightarrow k \text{ as } x \rightarrow c^+$$

The above-mentioned limit is right-sided and left sided limit can be presented as follows:

$$\text{Lim}_{x\text{-->}c^-} f(x) = k \text{ or } f(x) \rightarrow k \text{ as } x \rightarrow c^-$$

For example, consider a function f(x)=1/x. This function is not defined at x=0 and it is discontinuous at x=0. However, as x approaches 0 from $+\infty$ the value of function, f(x) approaches $+\infty$. The right-sided limit of a function can be written as:

$$\lim_{x\to 0^+} f(x) = +\infty$$

The left-sided limit of a function can be written as:

$$\lim_{x \to 0^-} f(x) = -\infty$$

The above function has different values from the right-side and left-side. Hence, this function is not well defined at x=0.

Let us consider a function $f(x) = \frac{1}{x}$

This function is undefined at x=0. The graph of this function is shown below:

Table 1: Right sided and left sided limits

Variable & Function	**Right-side limit**					**Right-side limit**			
X	1	0.6	0.4	0.3	0.0001	0.001	-0.08	-0.1	-0.4
F(x)	1	1.67	2.5	3.3	10000	-10000	-12.5	-10	-2.5

The graph of the above function is shown below:

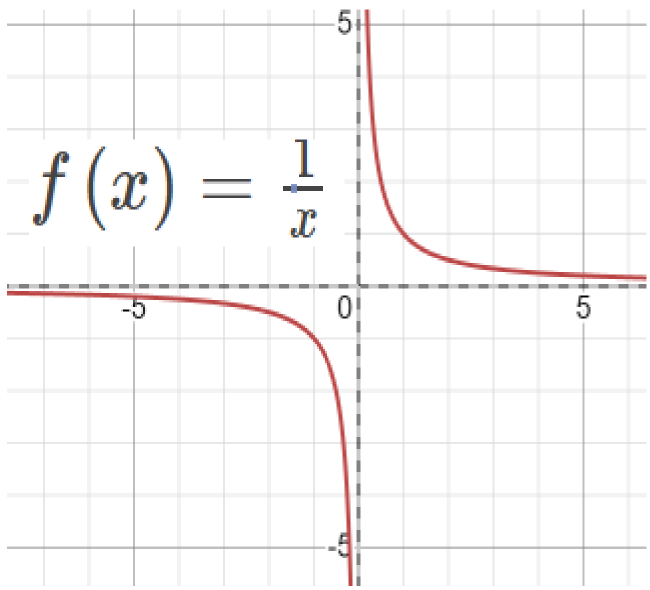


Figure 2: Graph of f(x)=1/x function

The above function, f(x)=1/x is discontinuous as x-→0 both from the left side as well as the right-side.

Similarly, consider another example $f(x) = x^3$. Here, also we can consider both left-side and right-side limits.

Table 2: Right-sided and left-sided limits of $f(x)= x^3$

	Right-side limit (x→+∞)					Left-side limit(x→-∞)				
X	0	1	5	10	100	-1	-5	-10	-100	-999
F(x)	0	1	125	1000	1000000	-1	-125	-1000	-1000000	-997002999

The plot of function $f(x)=x^3$ is shown in figure below:

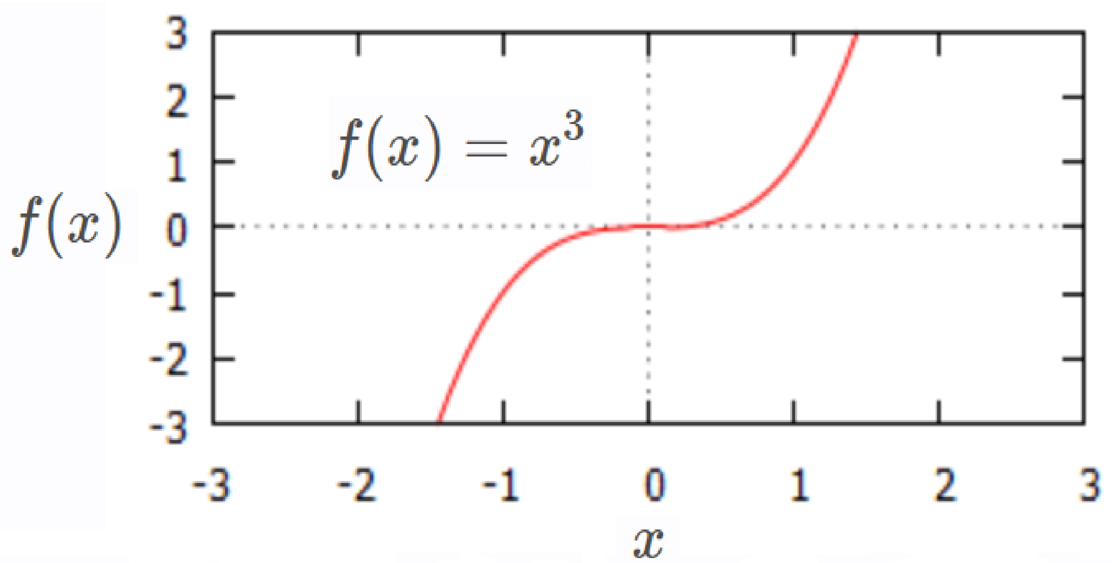


Figure 3: Graph of f(x)= x^3 function

It is evident from the above graph that as the value of x decreases from left, f(x) becomes more negative and as the value of x increases from right, it becomes more positive.

1.3.1 Properties of Limits

If the limits of two functions f(x) and h(x) exist, then the following properties can be applied as x⟶a:

1. Law of addition: $\lim_{x\to a} [f(x) + h(x)] = \lim_{x\to a} f(x) + \lim_{x\to a} h(x)$

2. Law of subtraction: $\lim_{x\to a} [f(x) - h(x)] = \lim_{x\to a} f(x) - \lim_{x\to a} h(x)$

3. Law of multiplication: $\lim_{x\to a} [f(x).h(x)] = \lim_{x\to a} f(x).$ limx $\lim_{x\to a} h(x)$

4. Law of Division: The division is applicable when limit of denominator is not equal to zero.

$$\lim x \to a \quad \frac{f(x)}{h(x)} = \frac{\lim x \to a \quad f(x)}{\lim x \to a \quad h(x)}$$

5. Law of root: The limit outside the root is applicable to the variable inside the root power

$$\lim_{x\to a} f(x) = \frac{\sqrt{\lim x\to a \quad x^3}}{c}$$ Here, c is a constant.

6. Law of power: According to the law of power, limit of a function raised to any power is equal to the limit of function raised to the same power.

$\lim_{x\to a} (f(x))^n = (\lim_{x\to a} f(x))^n$, where n is an integer.

7. Law of constant: According to law of constant, the limit of a constant function is equal to the constant value of the function. Let us suppose a constant function c, $\lim_{x\to a} f(c) = c$

8. Limit of constant multiplied function: If a function is multiplied with a constant has limit, it is equal to the limit value of function multiplied by the same constant:

$$\lim_{x\to a} c.f(x)= c. \lim_{x\to a} f(x)$$

9. Limits of trigonometric functions: The limits of trigonometric functions as lim $x\to \pm\infty$ is not defined. However, limits of trigonometric numbers, if defined has a specific value. For example, $\lim_{x\to a}$ Sin (x)=Sin a; $\lim_{x\to a}$ Cos (x)=Cos a; $\lim_{x\to a}$ tan (x)=tan a; $\lim_{x\to a}$ Cosec (x)=Cosec a; $\lim_{x\to a}$ Sec (x)=Sec a; $\lim_{x\to a}$ Cot (x)=Cot a.

10. Limits of inverse trigonometric functions: Inverse trigonometric functions are called as Arc functions. Arcsine ($\sin^{-1}(-x) = -\sin^{-1}(x)$, $x \in [-1, 1]$); Arccosine ($\cos^{-1}(-x) = \pi - \cos^{-1}(x)$, $x \in [-1, 1]$); Arctangent ($\tan^{-1}(-x) = -\tan^{-1}(x)$, $x \in R$); Arccotangent ($\cot^{-1}(-x) = \pi - \cot^{-1}(x)$, $x \in R$); Arcsecant ($\sec^{-1}(-x) = \pi - \sec^{-1}(x)$, $|x| \geq 1$); and Arccosecant ($\mathrm{cosec}^{-1}(-x) = -\mathrm{cosec}^{-1}(x)$, $|x| \geq 1$).

1.4 LIMIT OF A SEQUENCE

A sequence is the chronological arrangement of two or more things, either in finite or infinite form. A finite sequence has an ending term, but an infinite sequence has no ending term. For example, if the 1, 2, 3, 4, 5 sequence has five entries and the last entry is 5, it is a finite sequence; however, a sequence for example 1, 2, 3, 4, 5,... has infinite terms, called an infinite sequence. Each entry in a sequence is called a term. A series can be the sum of all items in a sequence. Sequences can be also be classified as arithmetic, geometric, harmonic, and fibonacci numbers. Arithmetic sequences are formed by subtracting or adding a specific number from the preceding number of the arithmetic sequence. On the other side, harmonic sequences have reciprocal terms with the arithmetic sequence. The geometric sequences are formed by dividing or multiplying a specific number with the preceding number of the geometric sequence. A sequence formed by adding two preceding elements that start with 0 and 1 is called a fibonacci number. The following are a few examples of these sequences:

1.4.1 Arithmetic Sequences

An arithmetic sequence can have terms like a, a + d, a + 2d. + a(n-1) d. Where n is the nth term of the sequence and d is the difference between the succeeding and preceding terms. Any term (a_n) in the arithmetic sequence can be found using $a_n = a+(n-1)\ d$. The sum of all terms $Sn = \frac{n(2a+(n-1)d)}{2}$ The limit of a sequence can also be calculated using limits as the sequence approaches a specific value. For example, we have a sequence 1, 3/2, 5/3, 7/4, 9/5, 11/6, 13/7…2-1/n.

$$\lim_{n\to\infty} (2-1/n) = 2$$

As n approaches ∞ the limit 1/n approaches zero, and the value of the limit function is equal to 2. However, this sequence does not have any term=2. Hence, it is evident that a limit may or may not contain a limit as a term in the sequence.

1.4.2 Geometric Sequences

A geometric sequence can assume the terms a, ar, ar^2, ar^3 , …$ar^{(n-1)}$. The limits of a geometric sequence can be calculated as shown in the following example:

Let us have an infinite series with sum G_∞ =1/2+ 1/4+ 1/8+1/16+1/32 … =1

The sum of this series can be calculated using the following formula:

$$G\infty = \frac{a(1-r^n)}{1-r}$$

If the value of r>1, r^n will grow exponentially as value of r increases to an infinitely large value and will decrease exponentially when r<1. So, in the limit notation, we can write when r<1 as:

$\lim_{n\to\infty} r^n = 0$ and can take limit value of $G\infty = \frac{a}{(1-r)}$

In a similar fashion, we can find limits to harmonic sequence and fibonacci numbers.

1.5 CONTINUITY OF FUNCTIONS

We can see continuity in many natural phenomena around us. For example, water is flowing in rivers, blood circulates in the human body, and growth of plants, etc. It is just like drawing a graph without lifting the pencil. A continuous function in mathematics holds the following properties at a point x = b:

1. f(b) has a finite value, and it exists.
2. $\lim_{x\to b} f(x)=f(b)$
3. The right-side and left-side limits are finite and equal.

A function would be continuous in an interval if all three conditions are met. There are two important theorems regarding the continuity of functions. These are discussed in the next section.

1.5.1 Intermediate Value Theorem

According to this theorem, a function f(x) is continuous in the interval [a, b], if there exists a minimum and maximum value in the interval [a, b] as follows:

$$a \leq x_{min} \leq b \text{ and } a \leq x_{max} \leq b$$

Also, the function f(x) has maximum $f(x_{max})$ and minimum $f(x_{min})$ values as follow:

$$f(x_{min}) \leq f(x) \leq f(x_{max}) \text{ when } a \leq x \leq b.$$

This theorem advocates that continuous functions always have minimum and maximum values.

1.5.2 Extreme Value Theorem

This theorem helps to find minimum and maximum values of a continuous function. According to this theorem, if a real-valued function is continuous in the closed interval [c, d] where c<d, then there exist two real numbers 'a' and 'b' in this closed interval [c, d] such that f(a) is the minimum and f(b) is the maximum value of the function f(x). Mathematically, it is shown as follows:

$$f(a) \leq f(x) \leq f(b), \forall x \in [a, b]$$

1.6 TYPES OF DISCONTINUITY

There are three different types of discontinuity: removable discontinuity, infinite discontinuity, and jump discontinuity. These are discussed as follows:

1.6.1 Removable Discontinuity

This type of discontinuity occurs when $\lim_{x \to a} f(x) \neq f(a)$. For example, show that the following function is not continuous:

$$f(x) = \frac{4x + 10}{x^2 - 2x - 15}$$

Now, we can factorise the denominator $x^2 - 2x - 15 =$ (x-5) (x+3). As we know, rational functions become discontinuous when divided by zero. The denominator shows that function is discontinuous when (x-5) (x+3) =0. It will happen when x=5 and/or x=-3.

Exercise 6: Show that the $f(k)= 2k^3 - 5k^2 - 10k + 5$ has a root within the interval range [-2, 2]

Solution: Here we need to check if f(k)=0 between interval [-2, 2]. Now suppose a number t between such that -2< t < 2 and f(k) = 0

The value of $f(-2) = 2(-2)^3 - 5(-2)^2 - 10(-2) + 5 = -16 - 20 + 20 + 5 = -11$

and $f(2) = 2(2)^3 - 5(2)^2 - 10(2) + 5 = 16 - 20 - 20 + 5 = -19$

Now, as per Intermediate Value theorem t point that shall exist between the interval -2 < t < 2

This solution did not provide us any such point. Now, we can use graphical method to locate this point as shown in figure below:

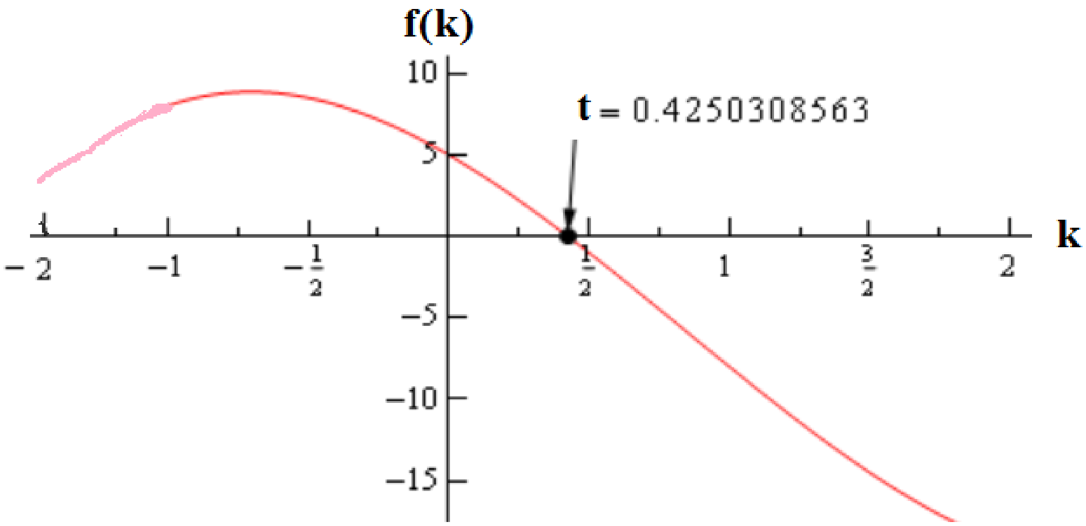


Figure 4: Graph of function f(k)= $2k^3 - 5k^2 - 10k + 5$

Let us take another example,

$$f(x) = \frac{(x+3)(x+1)}{(x+1)}$$

This function looks like f(x)=x+3. However, if we plot the curve, we will find that the function is not continuous at x=-1 which is evident from the following graph:

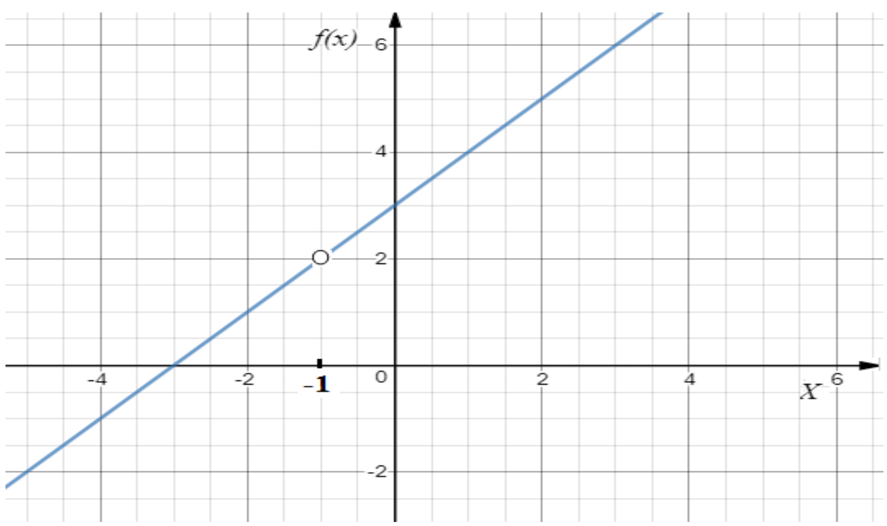


Figure 5: Plot of removable discontinuity graph

The hole in the graph is shown by the circle on the line. It shows the discontinuity of this function.

1.6.2 Infinite Discontinuity

It is also called as essential discontinuities due to severity of the infinite discontinuity points. This type of discontinuity arises when a function is not defined at a specific value, with the right-side and left-side limits tending to ±∞ values. Economists can classify it as large a scale discontinuity or a small-scale discontinuity. They can split this discontinuity to find a solution to their problem. For example, we have a function:

$$f(x) = \frac{1}{x}$$

With limit x→0 either from the right-side or from the left-side, the function is not defined at x = 0 which is evident from the graph below showing infinite discontinuity. The limit on the left side

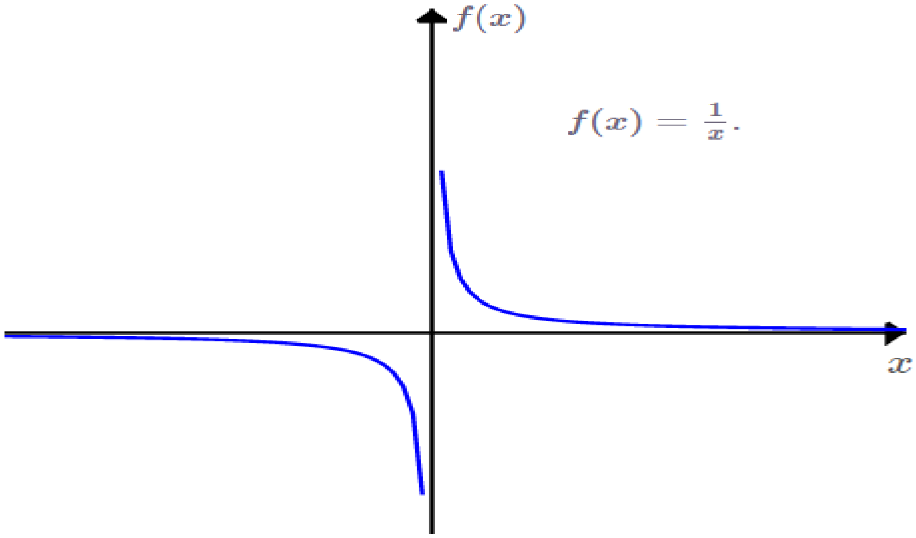


Figure 6: Graph of infinite discontinuity of function f(x)=1/x

of y-axis, as the value of x approaches 0, the infinitely smaller on the negative y-axis. Similarly, on the right-side of y-axis as x approaches zero, the function becomes infinitely large. The infinite discontinuity shall be visible if we the graph of function under evaluation. In this example, the function has infinite discontinuity at a value of x=0. It is called a pole in the graph.

1.6.3 Jump Discontinuity

This type of discontinuity is never infinite because the limits from both the right and left are real numbers. Let us consider the example given below:

$$f(x) = \begin{cases} 0, & x<0 \\ 2, & x\geq 0 \end{cases}$$

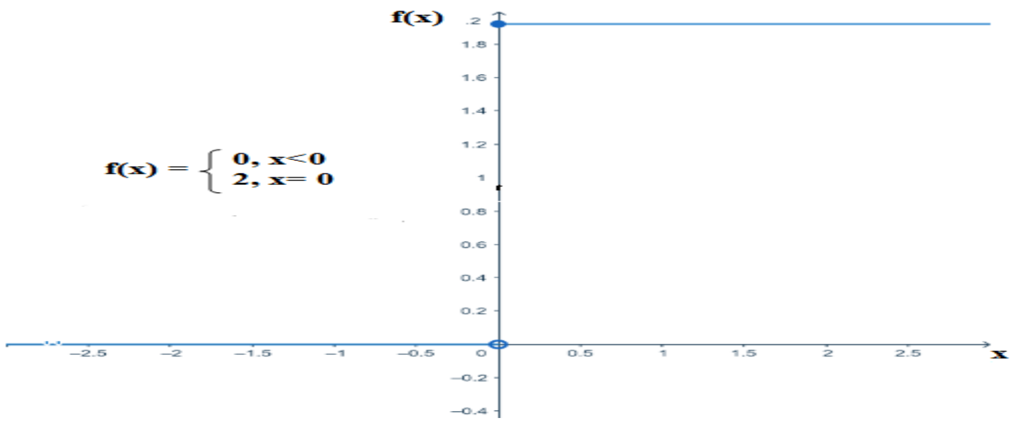


Figure 7: Graph of jump discontinuity

1.7 DIFFERENTIATION

Differentiation is the finding derivative of a function. It is used to know the instantaneous (very small interval) rate of change of any quantity. For example, velocity of a scooter tells the rate of change in distance over time. Differentiation in mathematics is defined as the rate of change of an independent variable.

If a function y = f(x) goes through an infinitesimal change at any point, then it can show as follows:

$$y=f(x)$$

$y+\Delta y= f(x+\Delta x)$, then limiting relative change is given by

$$\text{Lim } \Delta \rightarrow 0 \ \frac{(y+\Delta y)-y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x} = dy/dx$$

We need to note here that a differentiable function at any point is continuous at that point. However, a continuous function at a point may or may not be differentiable at that particular point. The following are important rules of differentiation:

Let the two functions $f : \mathbb{R} \rightarrow \mathbb{R}$ and $k : \mathbb{R} \rightarrow \mathbb{R}$ are differentiable functions at point x, then the following properties hold:

1. Differentiation of a polynomial: $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}$

$$\text{Given } f(x) = x^{-10} \Rightarrow f'(x) = -10x^{-10-1} = -10x^{-11}$$

2. Differentiation of Constant multiplier of a function: Let α be a constant multiplied to a function $\alpha f(x)$. The derivative $(\alpha f(x))' = \alpha f'(x)$, if $\alpha \in \mathbb{R}$ (constant).

$$\text{Given } f(x) = 3x^5 \Rightarrow f'(x) = 3 .5. x^{5-1} = 15x^4.$$

3. Differentiation of addition or subtraction of functions: The derivative of the sum or subtraction of two or more functions is equal to the sum or subtraction of their individual derivatives:

$$(f(x) \pm k(x))' = f'(x) \pm k'(x) \text{ (addition/subtraction).}$$

Given $f(x) = (x^2 \pm 3x^6)$:

$$f'(x) = f'(x^2) \pm f'(3x^6)$$
$$= 2x + 3.6\, x^{6-1}$$
$$= 2x \pm 18\, x^5$$

Given $f(x) = (x^3 + 3x^5)^4$:

$$\text{Let } g(x) = (x^3 + 3x^5) \Rightarrow g'(x) = 3x^2 + 15x^4$$
$$\therefore f'(x) = 4\,(x^3 + 3x^5)^{4-1}.\,(3x^2 + 15x^4)$$
$$= 4(x^3 + 3x^5)^3.\,(3x^2 + 15x^4)$$

4. Derivative of multiplication of two functions: Let *f* and *g* be two functions and their product is *fg*. The derivative of two multiplying functions is given by:

$$(f(x).\,k(x))' = f'(x)\,k(x) + f(x)\,k'(x)$$

Given $f(x) = (x)^3 . (5x)$:

$$\text{We know that } f'(x) = h(x).\,g'(x) + g(x).\,h'(x)$$
$$= x^3(5) + 5x(3x^2)$$
$$= 5x^3 + 15x^3$$
$$= 20x^3$$

5. Derivative of dividing functions: The derivative of division of the functions is given as:

$$\boldsymbol{derivative\,(f(x)/k(x)) = \frac{f'(x).k(x) - f(x).k'(x)}{square\ of\ k(x)}}$$

Given $f(x) = \frac{x+1}{x-2}$ $Where \neq 2$

As we know that $f'(x) = [h(x).\,g'(x) - g(x).h'(x)] / [h(x)]^2$

Let $s(x) = x + 1$ and $t(x) = x - 2$

Differentiate them individually we have. $s'(x) = 1$ and $t'(x) = 1$ and substitute the value in the formula we get the following:

$$\boldsymbol{f'(x) = \frac{1.(x-2) - (x-1).1}{(x-2).(x-2)}}$$

$$= \frac{x-2-x+1}{(x-2).(x-2)}$$

$$=-1/(x-2)^2$$

1.7.1 Types of Differentiation

There are five basic types of functions, where we can apply differentiation. These are trigonometric functions, algebraic functions, logarithmic functions, exponential functions, and mixed functions. The differentiation of these functions is discussed as follows:

Differentiation of trigonometric functions: The differentiation of trigonometric functions is shown below:

SN	Function f(x)	Differentiation f '(x)	SN	Function f(x)	Differentiation f '(x)
1.	sin *x*	cos *x*	7.	cos *x*	−sin *x*
2.	tan *x*	$\sec^2 x$	8.	cot *x*	$-\csc^2 x$
3.	sec *x*	sec *x* tan *x*	9.	csc *x*	−csc *x* cot *x*
4.	$\sin^{-1} x$	$\frac{1}{\sqrt{(1-x^2)}}$	10.	$\text{Cos}^{-1} x$	$-\frac{1}{\sqrt{(1-x^2)}}$
5.	$\tan^{-1} x$	$1/(1+x^2)$	11.	Cot^{-1}	$-1/(1+x^2)$
6.	$\sec^{-1} x$	$\frac{1}{IxI\sqrt{(1-x^2)}}$	12	$\text{Csc}^{-1} x$	$-\frac{1}{IxI\sqrt{(1-x^2)}}$

Exercise 7: Given f (x)=Cos x- Sin x, find its derivative.

Solution: Let f(x) = cos x; and g(x) = sin x

Using the subtraction rule of differentiation,

d/dx [f(x) – g(x)] = d/dx f(x) – d/dx g(x)

d/dx (cos x – sin x) = d/dx (cos x) – d/dx (sin x)

= - sin x – cos x

Differentiation of algebraic functions:

Differentiation Rule	f (x)	f '(x)
Sum & Difference	$\frac{d[f(x) \pm g(x)]}{dx}$	$\frac{d}{dx} f(x) \pm \frac{d}{dx} g(x)$
Product	$\frac{d[f(x).g(x)]}{dx}$	$g(x)\frac{d}{d(x)} f(x) + f(x)\frac{d}{d(x)} g(x)$

Quotient	$\frac{d[f(x)]}{g(x)}$	$\frac{g(x)\ \frac{f(x)}{dx} - f(x).\frac{g(x)}{dx}}{[g(x)]^2}$

Exercise 8: Given f (x)= 3x + (1/x), find its derivative.

Solution: Let us assume two parts of this function g(x) = x and h(x) = 1/x

According to sum rule of differentiation the derivative is equal to the sum derivative of individual terms. Hence,

d/dx [g(x) + h(x)] = d/dx g(x) + d/dx h(x)

d/dx [3x + (1/x)] = d/dx (3x) + d/dx (1/x)

= 3 + $(-1/x^2)$

= 3 – $(1/x^2)$

Exercise 9: Given f (x) = $(4x^3 – 6x + 1)$ (x + 1), find its derivative.

Solution: Let us assume two parts of this function g(x) = x and h(x) = 1/x

Let f(x) = $(4x^3 – 6x + 1)$ and g(x) = (x + 1).

According to the product rule of differentiation,

d/dx [f(x). g(x)] = f(x) [d/dx g(x)] + g(x) [d/dx f(x)]

= $(4x^3 – 6x + 1)$ [d/dx (x + 1)] + (x + 1) [d/dx $(4x^3 – 6x + 1)$]

= $(4x^3 – 6x + 1)$ (1 + 0) + (x + 1) [$4(3x^2)$ – 6(1) + 0]

= $(4x^3 – 6x + 1)$ + (x + 1) $(12x^2 – 6)$

= $4x^3 – 6x + 1 + 12x^3 – 6x + 12x^2 – 6$

= $16x^3 + 24x^2 – 12x – 5$

Exercise 10: Given f (x) = **$3x^2$/(x+1),** find **its derivative**

Solution: Here, f(x) = $3x^2$/ (x + 1)

g(x) = $3x^2$ ⇒g'(x) = 6x

h(x) = (x + 1) ⇒h'(x) = 1

According to the quotient rule of differentiation ⇒f '(x) = $[h(x)g'(x) - g(x)h'(x)]/[h(x)]^2$

⇒f ' (x) = $[(x+1).6x - 3x^2•1] / (x + 1)^2$

⇒f ' (x) = $(6x^2 + 6x - 3x^2) / (x + 1)^2$

⇒f'(x) = $(3x^2 + 6x)/(x + 1)^2$

Differentiation of logarithmic functions: Chain rule is used for differentiation logarithmic functions. Important logarithmic differentiation formulas are given below:

1. log xy =log x+ log y

2. Log x/y= log x - log y

log x^y = y log x

$\log_y x$ = log (x) / log (y)

Note: Derivative of exponential functions is calculated using logarithmic formulas.

Exercise 11: Given, y = x^x, find its derivative.

Solution: Take log on both sides: log(y) = log(x^x)

Using logarithmic rules ⇒ log(y) = x · log(x) [Using property log(a^x) = x· log(a)]

Step 3: Now differentiate the equation with respect to x ⇒

$$\frac{d}{dx}\log(y) = \frac{d}{dx}(x.\log(x))$$

$$\frac{\mathrm{d}}{\mathrm{dx}}\log(y) = x.\frac{d}{dx}\log(x) + \log(x).\frac{dx}{dx}$$

$$\frac{1}{y}\frac{\mathrm{dy}}{\mathrm{dx}} = x.\frac{1}{x} + \log(x)$$

$$\frac{\mathrm{dy}}{\mathrm{dx}} = y.(1 + \log(x))$$

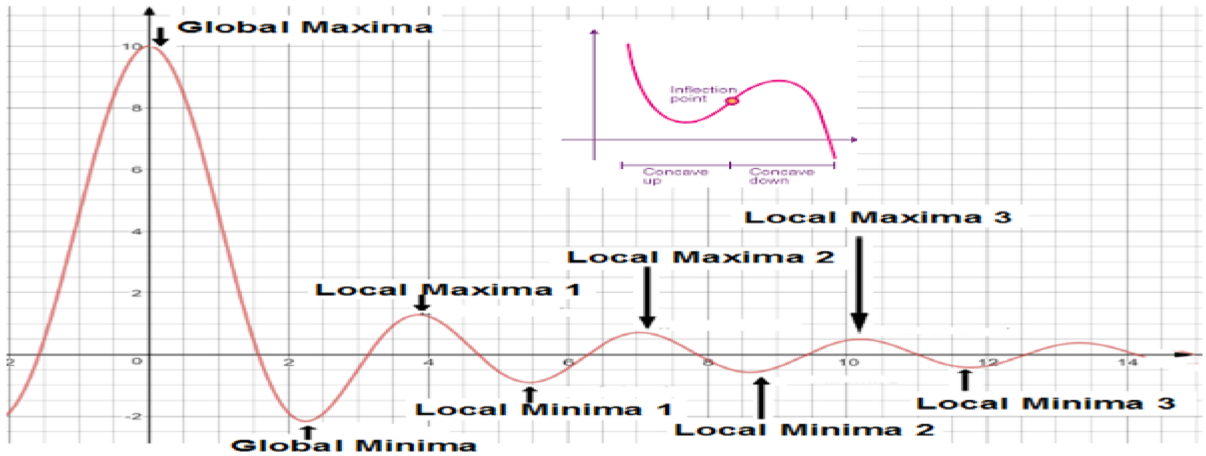
$$\frac{\mathrm{dy}}{\mathrm{dx}} = x^x(1 + \log(x))$$

1.8 MAXIMA AND MIMIMA

Maxima and minima show the maximum or minimum value of a function. The following procedure is adopted to find these two points:

1. Find the derivative of the given function.
2. Put this derivative = 0.
3. Find a second differentiation.
4. Put these values in the second derivative. If second derivative f''(x)<0, is the maxima, and if f''(x)>0, is the minima.
5. To find maximum and minimum values put the calculated values in the function f(x).

Maxima and minima are classified (**Figure 8**) as local maxima or minima and absolute or global maxima and minima. Local maxima or minima give the maximum or minimum value of a point when compared to other points in the domain of a function. The global maxima or minima is a point that doesn't have any points in the domain of the function whose value is greater or lesser than the value of the global maxima or minima. The point of inflexion shows the change from maxima to minima or vice versa.The knowledge of maxima and minima can help economists to know how profit functions limit the resources or expenditures.



(Source: Byjus and topper)

Figure 8: Maxima and minima of a function

Exercise 12: Let a function y = $3x^3 - 2x^2 + 8$, find maxima and minima.

Solution: Let f (x)=y= $3x^3 - 2x^2 + 8$

Calculate first derivative, f '(x)= $3.3\,x^2 - 2.2x^1 + 0$

$$=9x^2-5x=x\,(9x-5)$$

$$\text{Put } f'(x)=0=x(9x-5)=0$$

$$\Rightarrow \text{either } x=0 \text{ and or } 9x-5=0$$

$$9x=5 \Rightarrow x=5/9$$

Now we have two values of x where x=0 and x=5/9

Take second derivative and put the values

f " (x)= d $(9x^2-5x)/dx$ =18x-5

f " (x=0) =18x-5=18.0-5= -5 <0 (x=0 is point of maxima)

f"(x=5/9) = (18.5/9)-5=10-5=+5>0 (x=5/9 is point of minima)

Exercise 13: Let f(x)= $x^4 - 54x^2+11$, find point of inflexion.

Solution: f (x) = $x^4 - 54x^2+11$ (given)

The first derivative of the function f'(x) = $4x^3 - 108x$

The second derivative of the function f'' (x) = $12x^2 - 108$

Let us assume second derivative f'' (x) = $12x^2 - 108 = 0$

Divide by 12 on both sides, we get $x^2 - 9 = 0$

$\Rightarrow x^2 = 9$, Therefore, x = ± 3

To check inflexion at x = 3, substitute two adjoining points x= 2 and 4 in f’ (x)

So, f’ (2) = $12(2)^2 – 108 = -60$ (negative)

f’ (4) = $12(4)^2 – 108 = 84$ (positive)

To check for x = -3, substitute x= 0 and -4 in f’ (x)

So, f’ (0) = $12(0)^2 – 108 = -108$ (negative)

f’ (3) = $12(-4)^2 – 108 = 84$ (positive)

Hence, proved

Now, substitute x = ± 3 in f’ (x)

Therefore, it becomes

f’ (3) = $12(3)^2 – 108 = 0$

f’ (-3) = $12(-3)^2 – 108 = 0$

Therefore, the inflection points are (3, 0), and (-3, 0).

Exercise 14: A toy manufacturing company manufactures and sells 3000 units per month at a price of ₹100. Reducing the price by ₹ 5 shall help the company to sell 300 additional units per month. Predict the maximum possible revenue of the company.

Solution: Suppose that p is number of deductions of ₹ 5 from the base price of ₹ 100.

Then the unit price is 100 − 5p

The total amount of products sold for the month is 3000 + 300p

Hence the total revenue is given by (R) = (100 − 5p). (3000 + 300p)

$=300,000 + 30,000p - 15,000p - 1500p^2$

$= -1500p^2 +15,000p + 300,000$

Assume f (R)= $-1500p^2 +15,000p + 300,000$

The first differentiation of the revenue function f '(R)= -3000p+15000

Put f ' (R)=0

⇒3000p=15000

p=5, Now take the second derivative f "(R)

f "(R)=d (-3000p+15000)/dx   =-3000 <0

So, p= 5 is a point of maximum as the second derivative is -ve.

The maximum possible revenue per month=New price* No of units sold

= (100-5). (3300) = ₹3,13,500.

Increase in revenue= (100-5).3300-100.3000

=3,13,500-300000=₹13,500.

1.9 PARTIAL DIFFERENTIATION

The partial derivative of a function is used to take one tangent line of a given function to know its slope, is called partial differentiation. If a function f (x, y) depends on two independent variables x, y and we find its derivative with respect to x, keeping y as a constant, and its derivative with respect to y, keeping x as a constant, is called partial differentiation. It is shown as follows:

The partial derivative with respect to x while keeping y constant is given as:

$$f_x = \frac{\partial f}{\partial x} = lim_{k\to 0}\frac{f(x+k,y)-f(x,y)}{k}$$

The partial derivative with respect to y, keeping x as a constant, is given as:

$$f_{xy} = \frac{\partial f}{\partial y} = lim_{k\to 0}\frac{f(x,y+k)-f(x,y)}{k}$$

Rules of partial differentiation: The following rules are applicable to partial differentiation:

1. Multiplication rule of partial differentiation: Let we have a function f=g(x,y). h(x,y) . The value of partial differentiation of the product of these two functions g and h is give below:

$$f_x = \frac{\partial f}{\partial x} = g(x,y)\frac{\partial h}{\partial x} + h(x,y)\frac{\partial g}{\partial x}$$

$$f_y = \frac{\partial f}{\partial y} = g(x,y)\frac{\partial h}{\partial y} + h(x,y)\frac{\partial g}{\partial y}$$

2. Division or quotient rule: Let us assume a function $f = \frac{g(x,y)}{h(x,y)}$. Its partial derivative with respect to x and y is given below:

$$f_x = \frac{h(x,y)\frac{\partial g}{\partial x} - g(x,y)\frac{\partial h}{\partial x}}{[h(x,y)]^2}$$

$$f_y = \frac{h(x,y)\frac{\partial g}{\partial y} - g(x,y)\frac{\partial h}{\partial y}}{[h(x,y)]^2}$$

3.Power Rule of partial differentiation: Let f= $[g(x,y)]^n$, its partial derivative with respect to x and y is given below: **$f_x=n[g(x,y)]^{n-1}. \partial g/\partial x$**

$f_y=n[g(x,y)]^{n-1}. \partial g/\partial y$

4.Chain rule of independent and dependent variables: Let us have x=j(t) and y=k(t) are two single independent functions differentiable with respect to t. Also, there is another differentiable function z=f(x,y), differentiable with respect to x, and y. Now, we can write z=f(j(t), k(t)), which is differentiable with respect to t. The partial derivative of z with respect to 't' is given below:

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial t}$$

Let us have another functions x = j (u, v) and y = k (u, v) are two dependent differentiable functions and z = f (x, y) is a differentiable with respect to x and y. We can write z = f (j (u, v), k (u, v)). The partial derivative of z with respect to u and v is given below:

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}.\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}.\frac{\partial y}{\partial v}$$

The partial differentiation of logarithmic and trigonometric functions is done under the same procedure as discussed for normal differentiation.

Exercise 15: let f (x, y) = 4x + 3y, find its partial derivative with respect to x and y.

Solution: The partial derivative is calculated by differentiating the function with respect to one variable and keeping another variable as a constant.

$$f(x, y) = 4x + 3y$$

$$\frac{\partial \mathrm{f}}{\partial \mathrm{x}} = \frac{\partial(4x + 3y)}{\partial x} = \frac{\partial 4x}{\partial x} + \frac{\partial 3y}{\partial x} = 4 + 0 = 4$$

$$\frac{\partial \mathrm{f}}{\partial \mathrm{y}} = \frac{\partial(4x + 3y)}{\partial y} = \frac{\partial 4x}{\partial y} + \frac{\partial 3y}{\partial y} = 0 + 3 = 3$$

Exercise 16: Let f (x, y) = x^3y + sin x + cos y, find its partial derivative with respect to x and y.

Solution: We can differentiate this function with respect to one variable keeping all other variables constant

$$f_x = \partial f/\partial x = (3x)\, y + \cos x + 0$$

$$= 3xy + \cos x$$

Similarly,

$$f_y = \partial f/\partial y = x^3 + 0 + (-\sin y)$$

$$= x^3 - \sin y$$

1.10 SUGGESTED READINGS

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BACHELOR OF ARTS

SEMESTER – VI

QUANTITATIVE METHODS

UNIT 2 – MATRICES

STRUCTURE

2.0 OBJECTIVES

After studying the Unit, students will be able to:

- Define the Meaning and types of a Matrix
- Apply Addition, Subtraction and Multiplication of Matrices
- Apply the concept of Matrices to Business and Economic Problems
- Distinguish between Single and Double Entry Systems
- Find out Transpose a Matrix
- Understand the meaning of Symmetric, Skew-Symmetric, and Orthogonal Matrices.

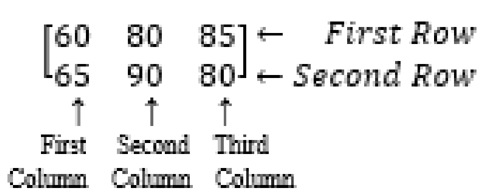
2.1 INTRODUCTION

Matrices are one of the most important and powerful business mathematics tools. Matrices applications help solve linear equations and with the help of this cost estimation, sales projection, etc. can be predicted. A matrix consists of a rectangular presentation of numbers arranged systematically in rows and columns describing the various aspects of a phenomenon inter-related in some manner.

For example: Marks obtained by two students, say, Ram and Shyam, in English, Mathematics, and Statistics are as follows:

	English	Mathematics	Statistics
Ram	60	80	85
Shyam	65	90	80

These marks may be represented by the following rectangular array enclosed by a pair of brackets



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Each horizontal line is called a row and each vertical line is called a column. The first row indicates the marks obtained by Ram in English, Mathematics and Statistics respectively and the second by Shyam in the three respective subjects.

Such a rectangular array is called a Matrix.

2.2 DEFINITION OF A MATRIX

An $m \times n$ matrix is a rectangular array of mn numbers (or elements) arranged in the form of an ordered set of m rows and n columns. A matrix A having m rows and n columns is typically written as

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m*n}$$

The horizontal lines are called rows and the vertical lines columns.

The numbers $a11, a12, \cdots,$ and amn belonging to the matrix are called elements.

A matrix having m rows and n columns is said to be of $order\ m \times n$ (read as 'm' by 'n').

The order may be written on the right of the matrix, as shown above.

2.3 NOTATIONS

Matrices are denoted by capital letters such as A, B, C, …, X, Y, Z and their elements by small letters a, b, c, … , $a11$, $a12$, etc. There are different nations enclosing the elements constituting a matrix in common use, viz., [ ], ( ), and { }, but we shall use the first one throughout the chapter. The suffixes of the element $a$ij depict that the elements lie in the ith row and jth column. Note that we always write the row number first and column number afterward. Also, note that, for the sake of brevity, the matrix A given above may also be written as A = [$a$ij] $m \times n$.

2.4 TYPES OF MATRICES

A. ROW MATRIX: If a matrix has only one row, it is called a $row\ matrix$. Thus, any 1 $\times n$ matrix is called a row matrix, for example,

$$A = [a1\ a2\ a3\ \ldots\ an]$$

is a row matrix of order $1 \times n$.

B. Column Matrix: A matrix consisting of only one column is called a columnatrix. In other words, any m × 1 matrix is called a column matrix, for example

$$A = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_m \end{bmatrix}$$

is a column matrix of order m × 1.

C. Zero or Null Matrix (0): If every element of an m × n matrix is zero, the matrix is called a zero matrix of order m × n, and is denoted by $0_{m \times n}$ or 0_{mn} or 0 simply. For example,

$$0_{2\times 3} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times 3}$$

is a zero matrix of order 2×3.

D. Square Matrix: Any matrix in which the number of rows is equal to the number of columns is called a Square matrix. Thus any $n \times n$ matrix is a square matrix of order n. Generally, we denote the order of a square matrix by a single number n, rather than $n \times n$.

Remark: The elements a_{ij} for which $iI = j$ in $A = [a_{ij}]_{n \times n}$ are called the diadonal lements and the line along which the elements $a_{11}, a_{12}, \dots, a_{nn}$ lie is called the leading diagonal or principal diagonal or diagonal simply. In a square matrix the pair of elements a_{ij} and a_{ji} are said to be conjugate elements.

E. Diagonal Matrix: A square in which all elements except those in the leading diagonal are zero, is called a diagonal matrix. Thus, a diagonal matrix of order n will be:

$$A = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Sometimes a diagonal matrix of order n with diagonal elements $a_{11}, a_{22}, \dots, a_{nn}$ is denoted by $A = \text{diag}\,(a_{11}, a_{22}, \dots, a_{nn})$.

F. Scalar Matrix: A diagonal matrix whose diagonal elements are all equal is called a scalar matrix. For example,

$$\begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

is a scalar matrix.

G. Identity or Unit Matrix (I): A scalar matrix in which each of its diagonal elements is unity is called an identity or unit matrix.

Thus, a square matrix $A = [a_{ij}]_{n \times n}$ is called identity matrix, if

$$a_{ij} = \begin{cases} 1, & \text{when } \; i = j, \\ 0, & \text{when } \; i \neq j. \end{cases}$$

An identity matrix of order n is denoted by I_n. Thus, $I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is a unit matrix of order 2.

H.Triangular Matrix: If every element above or below the leading diagonal is zero, the matrix is called a triangular matrix. If the zero element is lie below the leading diagonal, the matrix is called upper triangular matrix; If the zero elements is lie above the leading diagonal, the matrix is called lower triangular matrix. The matrices A_1 and A_2 given below are the examples of upper and lower triangular matrices respectively:

$$A_1 = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix} \text{(Upper triangular matrix)}$$

$$A_2 = \begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix} \text{(Lower triangular matrix)}$$

2.5 EQUALITY OF MATRICES

Two matrices are called comparable, if each of them consists of as many rows and columns as the other. Two matrices $A = [a_{ij}]$ and $B = [b_{ij}]$ are said to be equal, if (i) they have the same order and (iii) have equal corresponding elements throughout ($a_{ij} = b_{ij}$ for every i and j). The equality of matrices A and B.

Thus, the matrices $\begin{bmatrix} 1 & 7 \\ 3 & 5 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 9 \end{bmatrix}$ are not comparable while $\begin{bmatrix} 1 & 7 & 8 \\ 3 & 5 & 4 \end{bmatrix}$ and $\begin{bmatrix} 1 & 2 & 6 \\ 3 & 2 & 9 \end{bmatrix}$ are comparable but not equal.

The matrices $\begin{bmatrix} 1 & 5 & 9 \\ 3 & 4 & 12 \end{bmatrix}$ and $\begin{bmatrix} 1 & 5 & 3 \times 3 \\ 3 & 2 \times 2 & 3 \times 4 \end{bmatrix}$ are equal.

2.6 SUB MATRIX OF A MATRIX

A matrix which is obtained from a given matrix by deleting any number of rows or columns is called a sub-matrix of the given matrix.

For example, the matrix $\begin{bmatrix} 2 & 3 \\ 9 & 3 \end{bmatrix}$ is a sub-matrix of $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 9 & 3 \end{bmatrix}$. Obtained by deleting 2nd row and 1st column.

2.7 MULTIPLICATION OF A MATRIX BY A SCALAR

Let $A = [a_{ij}]$ be an $m \times n$ matrix and let k be any real number (called scalar). Then the product of k and A denoted by kA is defined to be the $m \times n$ matrix (i, j)th element is ka_{ij}, i.e.,

$$kA = \begin{bmatrix} ka_{11} & ka_{12} & \cdots & ka_{1n} \\ ka_{21} & ka_{22} & \cdots & ka_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ ka_{m1} & ka_{m2} & \cdots & ka_{mn} \end{bmatrix}$$

Thus, if $A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, 3A = \begin{bmatrix} 6 & 9 \\ 12 & 15 \end{bmatrix}$

Thus, we notice that, to get the scalar product each element of the given matrix is multiplied by the given scalar.

Properties of Scalar Multiplication:

(i) The product of a matrix with a scalar is commutative, i.e., $kA = Ak$.

(ii) If $k = -1, (-1)A = [-a_{ij}]$. Generally, $(-1)A$ is denoted by $-A$ and is called the negative of matrix A, Thus, $-[a_{ij}] = [-a_{ij}]$.

(iii) If A and B are comparable matrices and k is any scalar, we have $k(A + B) = kA + kB$.

(iv) If k and l are any two scalars and A is any matrix, we have $(k + l)A = kA + lA$.

(v) If k and l are any two scalars, we have $k(lA) = (kl)A$.

Example 1. Read the following elements $a_{21}, a_{32}, a_{22}, a_{11}$ in

$$A = \begin{bmatrix} 4 & 2 & 3 \\ 9 & 0 & 4 \\ 8 & 7 & 9 \end{bmatrix}$$

Solution: a_{21} denotes element in second row and first column,

$a_{21} = 9, a_{32} = 7, a_{22} = 0, a_{11} = 4.$

Example 2. Construct a 2×3 matrix $A = [a_{ij}]_{2\times3}$ whose general element is giving by

$$a_{ij} = (i - j)^2/2$$

Solution: $a_{11} = \frac{(1-1)^2}{2} = 0,\ a_{12} = \frac{(1-2)^2}{2} = \frac{1}{2},\ a_{13} = \frac{(2-1)^2}{2} = \frac{4}{2} = 2$

$a_{21} = \frac{(2-1)^2}{2} = \frac{1}{2},\ a_{22} = \frac{(2-2)^2}{2} = 0,\ a_{23} = \frac{(2-3)^2}{2} = \frac{1}{2}$

$$\therefore \quad A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{\mathbf{2\times3}}$$

$$\Rightarrow \quad A = \begin{bmatrix} 0 & \frac{1}{2} & 2 \\ \frac{1}{2} & 0 & \frac{1}{2} \end{bmatrix}_{\mathbf{2\times3}}$$

Example 3. If $A = \begin{bmatrix} 2 & 3 & 5 \\ 3 & -6 & 2 \end{bmatrix}$, find $-3A$.

Solution: $-3A = \begin{bmatrix} -3\times2 & -3\times3 & -3\times5 \\ -3\times3 & -3\times-6 & -3\times2 \end{bmatrix} = \begin{bmatrix} -6 & -9 & -15 \\ -9 & 18 & -6 \end{bmatrix}$

2.8 ADDITION OF MATRICES

The matrices A and B are conformable for addition, if they are comparable. i.e., B has the same number of rows and the same number of columns as A. Their sum, denoted by $A + B$, is defined to be the matrix obtained by adding the corresponding elements of A and B.

For example, if $A = \begin{bmatrix} 2 & 3 & 0 \\ 3 & 3 & 7 \end{bmatrix}$

and $B = \begin{bmatrix} 1 & 3 & 5 \\ 3 & 1 & 2 \end{bmatrix}$, we have

$$A + B = \begin{bmatrix} 2+1 & 3+3 & 0+5 \\ 3+3 & 3+1 & 7+2 \end{bmatrix}$$

$$A + B = \begin{bmatrix} 3 & 6 & 5 \\ 6 & 4 & 9 \end{bmatrix}$$

In general, if $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$, which is a matrix obtained by adding the elements in the corresponding positions. Thus, from

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m \times n} \quad \text{and}$$

$$B = \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{bmatrix}_{m \times n}$$

We get
$$A + B = \begin{bmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{bmatrix}_{m \times n}$$

2.9 DIFFERENCE OF MATRICES

If A and B are two comparable matrices, then their difference $A - B$ is matrix whose elements are obtained by subtracting the elements of B from the corresponding elements of A.

If $A = [a_{ij}]_{m \times n}$ and $B = [b_{ij}]_{m \times n}$

Then $A - B = [a_{ij} - b_{ij}]_{m \times n}$.

2.10 PROPERTIES OF MATRIX ADDITION

Suppose A, B, C are three matrices of the same order $m \times n$. Then the matrix addition has following properties:

A. Associativity

$$A + (B + C) = (A + B) + C$$

i.e., the addition of matrices is associative.

B. Commutativity

$$A + B = [a_{ij}] + [b_{ij}] = [a_{ij} + b_{ij}]$$

$$= [b_{ij} + a_{ij}] = [b_{ij}] + [a_{ij}]$$

$$= B + A$$

i.e., matrix addition is commutative.

C. Distributive law

$m(A + B) = mA + mB$ (m being an arbitrary scalar), because

$$m(A + B) = m[a_{ij} + b_{ij}]$$

$$= [m\,a_{ij} + m\,b_{ij}]$$

$$= m[a_{ij}] + m[b_{ij}] = mA + mB.$$

D. Existence of additive identity

Let A be any $m \times n$ matrix, and 0 the $m \times n$ null matrix. Then, we have

$$A + 0 = 0 + A = A$$

i.e., the null matrix is the identity for the matrix addition.

E. Existence of additive inverse

$-A$ is the additive inverse of A, because

$$(-A) + A = [-a_{ij}] + [a_{ij}]$$

$$= [-a_{ij} + a_{ij}] = 0 = A + (-A)$$

Thus, for any matrix A, there exists a unique additive inverse – A.

F. Cancellation law

$$A + B = A + C \Rightarrow [a_{ij} + b_{ij}] = [a_{ij} + c_{ij}]$$

$$\Rightarrow a_{ij} + b_{ij} = a_{ij} + c_{ij}$$

$$\Rightarrow b_{ij} = c_{ij}$$

$$\Rightarrow [b_{ij}] = [c_{ij}]$$

$$\Rightarrow B = C$$

This is said to be left cancellation.

Similarly, right cancellation, namely,

$B + A = C + A \Rightarrow B = C$ can be proved.

Example 4. If $A = \begin{bmatrix} 1 & 5 & 6 \\ -6 & 7 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & -5 & 7 \\ 8 & -7 & 7 \end{bmatrix}$, find $A + B$ and $A - B$.

Solution: Here $A + B = \begin{bmatrix} 1 & 5 & 6 \\ -6 & 7 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -5 & 7 \\ 8 & -7 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 1+1 & 5-5 & 6+7 \\ -6+8 & 7-7 & 0+7 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 0 & 13 \\ 2 & 0 & 7 \end{bmatrix}$$

and $A - B = \begin{bmatrix} 1 & 5 & 6 \\ -6 & 7 & 0 \end{bmatrix} - \begin{bmatrix} 1 & -5 & 7 \\ 8 & -7 & 7 \end{bmatrix}$

$$= \begin{bmatrix} 1-1 & 5-(-5) & 6-7 \\ -6-8 & 7-(-7) & 0-7 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 10 & -1 \\ -14 & 14 & -7 \end{bmatrix}$$

Example 5. If $A = \begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 3 \end{bmatrix}$, evaluate $3A - 4B$.

Solution: $3A - 4B = 3\begin{bmatrix} 2 & 3 & 1 \\ 0 & -1 & 5 \end{bmatrix} - 4\begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 6 & 9 & 3 \\ 0 & -3 & 15 \end{bmatrix} - \begin{bmatrix} 4 & 8 & -24 \\ 0 & -4 & 12 \end{bmatrix}$$

$$= \begin{bmatrix} 6-4 & 9-8 & 3-(-24) \\ 0-0 & -3-(-4) & 15-12 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 27 \\ 0 & 1 & 3 \end{bmatrix}.$$

Example 6. If X, Y are two matrices given by the equations

$$\mathbf{X + Y = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \text{ and } X - Y = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix}, \text{find } X, Y.}$$

Solution: We have

$$X + Y = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} \qquad \ldots(i)$$

$$X - Y = \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} \qquad \ldots(ii)$$

By adding equations (i) and (ii),

$$2X = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 3 & 2 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 1+3 & -2+2 \\ 3-1 & 4+0 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 2 & 4 \end{bmatrix}.$$

$\therefore$ $X = \frac{1}{2}\begin{bmatrix} 4 & 0 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix},$

From equation (i),

$$Y = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} - X = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 2 & 0 \\ 1 & 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1-2 & -2-0 \\ 3-1 & 4-2 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 2 & 2 \end{bmatrix}.$$

CHECK YOUR PROGRESS (A)

1. Find the elements $a_{31}, a_{24}, a_{34}, a_{22}$ in each of the following matrices given below. Also give their diagonal elements.

$$A = \begin{bmatrix} 8 & 6 & -3 & 1 \\ 2 & 3 & 5 & 0 \\ 3 & 9 & 5 & -7 \\ 5 & -3 & 1 & 0 \end{bmatrix}, \qquad A = \begin{bmatrix} 1 & 7 & 3 & 5 \\ 2 & 3 & -1 & 0 \\ 3 & 5 & 6 & 8 \\ 4 & 3 & 0 & 0 \\ 2 & 1 & 9 & 8 \end{bmatrix}$$

2. Write the matrix $A = [a_{ij}]$ of order 2×3 whose general elements is given by (i) $a_{ij} = ij$ (ii) $a_{ij} = (-1)^{ij}(i + j)$

3. Find x and y, if

$$\begin{bmatrix} x+y & z \\ 1 & x-y \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 1 & 7 \end{bmatrix}$$

4. If $A = \begin{bmatrix} 0 & 2 & 3 \\ 2 & 1 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 6 & 3 \\ 1 & 4 & 5 \end{bmatrix}$, find the value of $2A + 3B$.

5. Given $A = \begin{bmatrix} 1 & 2 & -3 \\ 5 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & -1 & 2 \\ 4 & 2 & 5 \\ 2 & 0 & 5 \end{bmatrix}$, find the matrix C such that $A + 2C = B$.

6. Solve the following equations for A and B:

$$2A - B = \begin{bmatrix} 3 & -3 & 0 \\ 3 & 3 & 2 \end{bmatrix}, 2B + A \begin{bmatrix} 4 & 1 & 5 \\ -1 & 4 & -4 \end{bmatrix}$$

ANSWERS

1. 3, 0, -7, 3 and 3, 0, 8, 3; the diagonal elements are 8, 3, 8, 0 and 1, 3, 6, 0.

2. (i) $\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \end{bmatrix}$ (ii) $\begin{bmatrix} -2 & 3 & -4 \\ 3 & 4 & 5 \end{bmatrix}$

3. $x = 5, y = -2$

4. $\begin{bmatrix} 21 & 22 & 15 \\ 7 & 14 & 23 \end{bmatrix}$

5. $\begin{bmatrix} 1 & -\frac{3}{2} & \frac{5}{2} \\ -\frac{1}{2} & 1 & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} & -2 \end{bmatrix}$

6. $A = \begin{bmatrix} 2 & -1 & 1 \\ 1 & 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 1 & 2 \\ -1 & 1 & -2 \end{bmatrix}$

2.11 MULTIPLICATION OF MATRIX

The multiplication of one matrix by another is possible, if and only if the number of columns of first matrix is equal to the number of rows of the second matrix. The resulting matrix will have the number of rows equal to those in the first matrix and the number of columns equal to those in the second matrix. Thus, if matrix A is of order $m \times n$ and B is of order $n \times p$, the product AB is possible, $\mathrm{i.e.}$, the matrices A and B are conformable for multiplication in the order A, B. The order of the resulting matrix AB will be $m \times p$. The (i,k)th element ($\mathrm{i.e.}$, the element laying ith row and kth column) of AB is given by

$$a_{i1}b_{1k} + a_{i2}b_{2k} + \cdots + a_{in}b_{nk} = \sum_{j=1}^{n} a_{ij}b_{jk}.$$

Thus, to obtain the (i,k)th element of the product AB, we multiply the elements of the ith row of A to the corresponding elements of the kth column of B and add the products thus obtained. The resulting sum is the (i,k)th element of AB.

If AB is denoted by $C = [c_{ij}]$, i.e., $AB = [c_{ij}]$, we have

$$\begin{bmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{in} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} b_{11} & \cdots & b_{ik} & \cdots & b_{ip} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{j1} & \cdots & b_{jk} & \cdots & b_{jp} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n1} & \cdots & b_{nk} & \cdots & b_{np} \end{bmatrix}$$

$$= \begin{bmatrix} c_{11} & \cdots & c_{1k} & \cdots & c_{1p} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{i1} & \cdots & c_{ik} & \cdots & c_{ip} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ c_{m1} & \cdots & c_{mk} & \cdots & c_{mp} \end{bmatrix}$$

Where $c_{ij} = \sum_{j=1}^{n} a_{ij}b_{jk}$. Putting $i = 1, 2, \cdots, m$ and $k = 1, 2, 3, \ldots, p$, all the elements of C will be found.

In the product AB, A is said to be pre-multiplier or pre-factor while B is said to be post-multiplier or post-factor. It is to be noted that in multiplying one matrix by another, unlike ordinary numbers, the placement of matrices as pre factor and post factor is very important. Thus AB is not the same as BA.

2×2 Matrix Multiplication Formula

Let us consider two matrices A and B of order “2 × 2”. Then its multiplication is achieved using the formula.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} and B = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

$$AB = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} p & q \\ r & s \end{bmatrix}$$

$$AB = \begin{bmatrix} (ap + br) & (aq + bs) \\ (cp + dr) & (cq + ds) \end{bmatrix}$$

3×3 Matrix Multiplication Formula

Let us consider two matrices P and Q of order “3 × 3”. Now, the matrix multiplication formula of “3 × 3” matrices is,

$$X = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix}$$

$$Y = \begin{bmatrix} y_{11} & y_{12} & y_{13} \\ y_{21} & y_{22} & y_{23} \\ y_{31} & y_{32} & y_{33} \end{bmatrix}$$

$$XY = \begin{bmatrix} (x_{11}y_{11} + x_{12}y_{21} + x_{13}y_{31}) & (x_{11}y_{12} + x_{12}y_{22} + x_{13}y_{32}) & (x_{11}y_{13} + x_{12}y_{23} + x_{13}y_{33}) \\ (x_{21}y_{11} + x_{22}y_{21} + x_{23}y_{31}) & (x_{21}y_{12} + x_{22}y_{22} + x_{23}y_{32}) & (x_{21}y_{13} + x_{22}y_{23} + x_{23}y_{33}) \\ (x_{31}y_{11} + x_{32}y_{21} + x_{33}y_{31}) & (x_{31}y_{12} + x_{32}y_{22} + x_{33}y_{32}) & (x_{31}y_{13} + x_{32}y_{23} + x_{33}y_{33}) \end{bmatrix}$$

2.12 PROPERTIES OF MATRIX MULTIPLICATION

1. Matrix multiplication is not commutative. To verify the above statement, let us take an example. Consider the matrices.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

It can easily find that

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \text{ while } BA = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

so that $AB \neq BA$.

This shows that matrix multiplication is not commutative. Actually speaking, for a given pair of matrices A and B, the products AB and BA may not be even comparable. For example, if A is an $m \times n$ matrix and B is an $n \times m$ matrix, AB would be an $m \times m$ matrix and BA would be an $n \times n$ matrix.

It may also happen that for a pair of matrices A and B the product AB may be defined but the product BA may not be defined. For example, it A is an $m \times n$ matrix and B is an $n \times p$ matrix, AB would be an $m \times p$ matrix, but it is not meaningful to talk of BA unless $m = p$.

Note 1. It is worthwhile to note that the statement 'matrix multiplication is not commutative' does not mean that there are no matrices A and B such that $AB = BA$. It simply means that generally $AB \neq BA$. Thus, we wish to convey that there do exist some pairs of matrices A and B for which $AB = BA$.

Note 2. It is also to be noted that in matrices, $AB = 0$ need not always imply that either $A = 0$ or $B = 0$, This will be clear, if we consider the matrices,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}.$$

For these matrices, $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, but none of A and B is zero matrix.

Note 3. The familiar cancellation law of multiplication for numbers fails to be true for matrix multiplication.

Below we give the properties which hold good for matrices.

2. **Associative law:** Let A, B and C be the matrices of suitable size for the products $A(BC)$ and $(AB)C$ to exist. Then, $A(BC) = (AB)C$.
3. **Distributive law:** $A(B + C) = AB + AC$, (left distributive) and $(B + C)D = BD + CD$ (right distributive), provided that the matrices A, B, C and D are of the sizes that they are conformable for the operations involved so that the above relations are meaningful.
4. **Multiplication of a matrix by a unit matrix:** If A is square matrix of order $n \times n$ and I is the unit matrix of the same order, we get

   $AI = A = IA$.

5. Multiplication of a matrix by itself: The product A, A is defined, if the number of column is equal to the number of rows of A, i. e., if A is a square matrix and in that case A. A $= A^2$ will also be a square matrix of the same type. Also,

$$A.A.A = A^2A = A^3.$$

Similarly, A. A. A .... n times$= A^n$.

Note: If I is a unit matrix, we have $I = I^2 = I^3 = \cdots = I^n$.

Example 7. If $A = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}$

and $B = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}$, Obtain the product AB and BA and show than AB $\neq$ BA.

Solution:

$$AB = \begin{bmatrix} 2 & 2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}_{3\times 3} \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}_{3\times 3}$$

Number of columns of A = Number of rows of B

∴ AB is defined

$$AB = \begin{bmatrix} 0+0+3 & 0+2+6 & 4+4+0 \\ 0+0-1 & 0+3-2 & 4+6+0 \\ 0+0+2 & 0+1+4 & -6+2+0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 8 & 8 \\ -1 & 1 & 10 \\ 2 & 5 & -4 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 1 & 2 \\ 1 & 2 & 0 \end{bmatrix}_{3\times 3} \begin{bmatrix} 2 & 2 & 3 \\ 2 & 3 & -1 \\ -3 & 1 & 2 \end{bmatrix}_{3\times 3}$$

$$= \begin{bmatrix} 0+0-6 & 0+0+2 & 0+0+4 \\ 0+2-6 & 0+3+2 & 0-1+4 \\ 2+4+0 & 2+6+0 & 3-2+0 \end{bmatrix}_{3\times 3}$$

$$= \begin{bmatrix} -6 & 2 & 4 \\ -4 & 5 & 3 \\ 6 & 8 & 1 \end{bmatrix}$$

Hence AB $\neq$ BA.

Example 8. Write down the products AB and BA of the two matrices A and B, where,

$$\mathbf{A} = [\mathbf{1} \quad \mathbf{2} \quad \mathbf{3} \quad \mathbf{4}] \text{ and } \mathbf{B} = \begin{bmatrix} \mathbf{1} \\ \mathbf{2} \\ \mathbf{3} \\ \mathbf{4} \end{bmatrix}$$

Solution: Since **A** is a 1×4 matrix and **B** is a 4×1 matrix. **AB** will be a 1×1 matrix.

$$\mathbf{AB} = [1 \quad 2 \quad 3 \quad 4] \times \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$= [1.1 + 2.2 + 3.3 + 4.4] = [30]$$

BA will be a 4×4 matrix.

$$\mathbf{BA} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \times 1 & 2 \times 1 & 3 \times 1 & 4 \times 1 \\ 1 \times 2 & 2 \times 2 & 3 \times 2 & 4 \times 2 \\ 1 \times 3 & 2 \times 3 & 3 \times 3 & 4 \times 3 \\ 1 \times 4 & 2 \times 4 & 3 \times 4 & 4 \times 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 9 & 12 \\ 4 & 8 & 12 & 16 \end{bmatrix}$$

Example 9. Evaluate $\mathbf{A^2 - 4A - 5I}$, where

$$\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} & \mathbf{2} \\ \mathbf{2} & \mathbf{1} & \mathbf{2} \\ \mathbf{2} & \mathbf{2} & \mathbf{1} \end{bmatrix} \textbf{ and } \mathbf{I} = \begin{bmatrix} \mathbf{1} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$$

Solution: We have

$$A^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

$$A^2 - 4A - 5I = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - 4 \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - 5 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9-4-5 & 8-8+0 & 8-8+0 \\ 8-8+0 & 9-4-5 & 8-8+0 \\ 8-8+0 & 8-8+0 & 9-4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0.$$

Where 0 is the null matrix.

Example 10. If $\mathbf{A} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$ and $\mathbf{I} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$, show that

$$\mathbf{A^2 - (a + d)A = (bc - ad)I.}$$

Solution: $\mathbf{A^2} = \mathbf{A.A} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \times \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix}$$

$$A^2 - (a + d)A = \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} - (a + d) \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$= \begin{bmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{bmatrix} - \begin{bmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{bmatrix}$$

$$= \begin{bmatrix} bc - ad & 0 \\ 0 & bc - ad \end{bmatrix} = (bc - ad) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= (bc - ad)\mathbf{I}.$$

Example 11. If $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{2} \\ \mathbf{3} & \mathbf{4} \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} \mathbf{1} & \mathbf{0} \\ \mathbf{2} & \mathbf{-3} \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{0} & \mathbf{1} \end{bmatrix}$, show that $\mathbf{A(B + C) = AB + AC}$.

Solution: We have

$$\mathbf{B + C} = \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 + 1 & 0 - 1 \\ 2 + 0 & -3 + 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix}.$$

$$\mathbf{A(B + C)} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 2 & -1 \\ 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 + 4 & -1 - 4 \\ 6 + 8 & -3 - 8 \end{bmatrix} = \begin{bmatrix} 6 & -5 \\ 14 & -11 \end{bmatrix}, \quad \ldots(i)$$

Again, $\mathbf{AB} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 2 & -3 \end{bmatrix}$

$$= \begin{bmatrix} 1 + 4 & 0 - 6 \\ 3 + 8 & 0 - 12 \end{bmatrix} = \begin{bmatrix} 5 & -6 \\ 11 & -12 \end{bmatrix}$$

and $\mathbf{AC} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1 + 0 & -1 + 2 \\ 3 + 0 & -3 + 4 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix},$$

$\therefore$ $\mathbf{AB + AC} = \begin{bmatrix} 5 & -6 \\ 11 & -12 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -5 \\ 14 & -11 \end{bmatrix} \quad \ldots(ii)$

From (i) and (ii), we have

$$\mathbf{A(B + C) = AB + AC.}$$

Example 12. $\mathbf{A} = \begin{bmatrix} \mathbf{1} & \mathbf{-1} \\ \mathbf{2} & \mathbf{-1} \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} \mathbf{a} & \mathbf{1} \\ \mathbf{b} & \mathbf{-1} \end{bmatrix}$ and $\mathbf{(A + B)^2 = A^2 + B^2}$ find a and b.

Solution: We have

$$A^2 = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} \times \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 1-2 & -1+1 \\ 2-2 & -2+1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

and $$B^2 = \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} \times \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{bmatrix}$$

∴ $$A^2 + B^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} a^2+b & a-1 \\ ab-b & b+1 \end{bmatrix} = \begin{bmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{bmatrix} \qquad \ldots(1)$$

Also $$A + B = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} a & 1 \\ b & -1 \end{bmatrix} = \begin{bmatrix} 1+a & -1+1 \\ 2+b & -1-1 \end{bmatrix} = \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix}$$

∴ $$(A+B)^2 = \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix} \times \begin{bmatrix} 1+a & 0 \\ 2+b & -2 \end{bmatrix} = \begin{bmatrix} (1+a)^2+0 & 0+0 \\ (2+b)(1+a)-2(2+b) & 0+4 \end{bmatrix}$$

$$= \begin{bmatrix} (1+a)^2 & 0 \\ (2+b)(a-1) & 4 \end{bmatrix}. \qquad \ldots(2)$$

Now given that $(A+B)^2 = A^2 + B^2$.

Hence from (1) and (2), we get

$$\begin{bmatrix} (1+a)^2 & 0 \\ (2+b)(a-1) & 4 \end{bmatrix} = \begin{bmatrix} a^2+b-1 & a-1 \\ ab-b & b \end{bmatrix}$$

or $a - 1 = 0$ and $b = 4$.

Hence $a = 1, b = 4$.

2.13 APPLICATIONS OF MATRICES TO BUSINESS AND ECONOMIC PROBLEMS

Matrices play a crucial role in various business and economic applications due to their ability to represent and solve complex systems of equations, model relationships between variables, and analyze data. Here are some key applications of matrices in business and economics:

1. Linear Programming Matrices are extensively used in linear programming, a mathematical technique for optimizing resource allocation. Businesses can use matrices to represent constraints, objective functions, and decision variables, allowing them to find the optimal solution for maximizing profits or minimizing costs.
2. Input-Output Analysis: Input-output models in economics use matrices to represent the relationships between different sectors of an economy. These models help analyze the interdependencies between industries, consumption, and production, aiding in policy decisions and economic planning.
3. predicting future economic conditions: Matrices is useful in predicting future economic conditions, such as consumer behavior, investment decisions, and market trends.
4. Financial Analysis: Matrices are used in finance to model and analyze investment portfolios.

The risk and return associated with different assets can be represented in matrix form, facilitating the calculation of portfolio diversification and optimization.

6. Supply Chain Management: Matrices can represent the flow of goods, information, and resources within a supply chain. This allows businesses to optimize logistics, minimize costs, and enhance efficiency in the production and distribution processes.
7. Network Analysis: Matrices are used to model and analyze networks, such as social networks or transportation networks. This can be applied to understand the flow of information, goods, or services within a network, aiding in decision-making for resource allocation and optimization.
8. Time Series Analysis: Matrices are used to represent and analyze time series data in economics. Techniques like matrix decomposition can help identify trends, seasonal patterns, and underlying structures in economic time series
9. Risk Management: Matrices are used to model and analyze risk in financial and business contexts. For example, covariance matrices can be employed to assess the risk associated with different assets in a portfolio.

These applications highlight the versatility of matrices in addressing various quantitative aspects of business and economic problems, making them a valuable tool for decision-makers and analysts.

Example 14. A manufacturer produces three items P, Q and R and sells them in two markets I and II. Annual sales are given below:

	P	**Q**	**R**
I	**6,000**	**2,000**	**3,000**
II	**8,000**	**4,000**	**2,000**

If sales price of each unit of P, Q, R is Rs. 4, Rs. 3, Rs. 2 respectively, then find the total revenue of each market using matrix.

Solution: Let $A = \begin{matrix} \text{I} \\ \text{II} \end{matrix}\begin{bmatrix} 6{,}000 & 2{,}000 & 3{,}000 \\ 8{,}000 & 4{,}000 & 2{,}000 \end{bmatrix}$ is the sales matrix and

$$B = \begin{matrix} \text{P} \\ \text{Q} \\ \text{R} \end{matrix}\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} \text{ is the price matrix}$$

$\therefore$ Revenue matrix $AB = \begin{bmatrix} 6{,}000 & 2{,}000 & 3{,}000 \\ 8{,}000 & 4{,}000 & 2{,}000 \end{bmatrix}\begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} 6{,}000 \times 4 & + & 2{,}000 \times 3 & + & 3{,}000 \times 2 \\ 8{,}000 \times 4 & + & 4{,}000 \times 3 & + & 2{,}000 \times 2 \end{bmatrix} = \begin{bmatrix} 36{,}000 \\ 48{,}000 \end{bmatrix}$$

Hence

Total Revenue of Market I = Rs. 36,000

Total Revenue of Market II = Rs. 48,000

CHECK YOUR PROGRESS (B)

1. If $\mathbf{A} = [3 \quad 6 \quad -5]$ and $\mathbf{B} = \begin{bmatrix} 4 \\ 7 \\ 2 \end{bmatrix}$, find **AB** and **BA.**

2. If $\mathbf{A} = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 1 & -1 \\ -3 & 2 \end{bmatrix}$, find **AB** and **BA.** Is $=$ **BA** ?

3. If $\mathbf{A} = \begin{bmatrix} 1 & -2 & 3 \\ -4 & 2 & 5 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 2 & 3 \\ 4 & 5 \\ 2 & 1 \end{bmatrix}$, find **AB** and show that $\mathbf{AB} = \mathbf{BA}.$

4. When $\mathbf{A} = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} i & -1 \\ -1 & -i \end{bmatrix}$ and $i = \sqrt{(-1)}$, determine **AB**. Also compute **BA**.

5. If $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 4 \\ 3 & -2 & 3 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} -1 & -1 & 1 \\ 2 & 2 & -1 \\ -3 & 3 & 3 \end{bmatrix}$,

Show that **AB** and **CA** are the null matrices but **BA** and **AC** are not the null matrices.

6. If $\mathbf{A} = \begin{bmatrix} 3 & 2 \\ 4 & 1 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} a & b \\ 3 & 5 \end{bmatrix}$,

   Find a and b such that $\mathbf{AB} = \mathbf{BA}.$ Also compute $\mathbf{3A} + \mathbf{5B}.$

7. If $\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 5 & 1 \\ 7 & 4 \end{bmatrix}$, show that $\mathbf{A}(\mathbf{B} + \mathbf{C}) = \mathbf{AB} + \mathbf{AC}$.

8. If $\mathbf{A} = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$, prove by mathematical induction that $\mathbf{A^n} = \begin{bmatrix} 1 & na \\ 0 & 1 \end{bmatrix}$

9. Given $\mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 0 & 3 \\ 3 & -1 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \\ -1 & 4 \end{bmatrix}$ and $\mathbf{C} = \begin{bmatrix} 1 & 2 & 3 & -4 \\ 2 & 0 & -2 & 1 \end{bmatrix}$,

   Show that $(\mathbf{AB})\mathbf{C} = \mathbf{A}(\mathbf{BC})$.

10. If $A = \begin{bmatrix} -5 & -2 & 1 \\ 4 & 3 & 3 \\ -6 & 6 & -2 \end{bmatrix}$

Find the matrix B such that $A + B =$ unit matrix.

11. If $\begin{bmatrix} 1 & 1 & x \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 0$ **find x**

12.There are two families A and B. In family A, there are 4 men, 6 women and 2 children; and in family B there are 2 men, 2 women and 4 children. The recommended daily requirement for calories is:

Calories:	Man 2,400;	Woman 1,900;	Child 1,800
Protein:	Man 55 gm;	Woman 45 gm;	Child 33 gm

Calculate the total requirements of calories and proteins for each of the two families using matrix method.

13.The co-operative store of a particular school has 10 dozen books of physics, 8 dozen of chemistry books and 5 dozen of mathematics books. Their selling price are Rs. 65.70, Rs. 43.20 and Rs. 76.50 respectively. Find by matrix method the total amount, the store will receive from selling all three items.

Answers

1. $\mathbf{AB} = [44]$, $\mathbf{BA} = \begin{bmatrix} 12 & 24 & -20 \\ 21 & 42 & -35 \\ 6 & 12 & -10 \end{bmatrix}$,

2. $\mathbf{AB} = \begin{bmatrix} -13 & 8 \\ -8 & 5 \end{bmatrix}$, $\mathbf{BA} = \begin{bmatrix} 1 & 2 \\ -4 & -9 \end{bmatrix}$,

3. $\mathbf{AB} = \begin{bmatrix} 0 & -4 \\ 10 & 3 \end{bmatrix}$

4. $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, $BA = \begin{bmatrix} 2i & -2 \\ -2 & -2i \end{bmatrix}$,

6. $a = 65, b = 15, \begin{bmatrix} 41.5 & 13.5 \\ 27 & 28 \end{bmatrix}$

10. $\begin{bmatrix} 6 & 2 & -1 \\ -4 & -2 & -3 \\ 6 & -6 & -1 \end{bmatrix}$

11. -2

12. Calories for family A and B are 24,600 and 15,800 and proteins are 556 gms and 332 gms respectively.

13. Rs. 16,621.20

2.14 TRANSPOSE OF A MATRIX

A matrix obtained by interchanging the corresponding rows and columns of a given matrix A is called the **transpose matrix** of **A.** The transpose of a matrix is denoted by A^T or A'.

For example,

If $A = \begin{bmatrix} 1 & 5 \end{bmatrix}$, then $A' = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$

If $A = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$, then $A' = [2 \quad -3]$

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, then $A' = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

Properties of the transpose matrix

(i) $(A')' = A$

(ii) $(A + B)' = A' + B'$

(iii) $(\lambda A)' = \lambda A'$

(iv) If A and B are two matrices which are conformable for multiplication, then

$$(AB)' = B'A'$$

This is called 'Reversal Law'.

Example 15: If $A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}$

and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$

Then, verify that $(AB)^T = B^T A^T$

Solution: $A = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix}_{3\times 3}$ and $B = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}_{3\times 3}$

$$AB = \begin{bmatrix} 1 & 2 & -3 \\ 3 & 0 & 2 \\ 4 & 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} (1+4+0) & (0+2-3) & (0+0-9) \\ (3+0+0) & (0+0+2) & (0+0+6) \\ (4+10+0) & (0+5+0) & (0+0+0) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & -1 & -9 \\ 3 & 2 & 6 \\ 14 & 5 & 0 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 5 & 3 & 14 \\ -1 & 2 & 5 \\ -9 & 6 & 0 \end{bmatrix} \qquad \ldots(i)$$

$$A^T = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -3 & 2 & 0 \end{bmatrix}_{3\times 3}, B^T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}_{3\times 3}$$

$$B^T A^T = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 4 \\ 2 & 0 & 5 \\ -3 & 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} (1+4-0) & (3+0+0) & (4+10+0) \\ (0+2-3) & (0+0+2) & (0+5+0) \\ (0+0-9) & (0+0+6) & (0+0+0) \end{bmatrix}$$

$$= \begin{bmatrix} 5 & 3 & 14 \\ -1 & 2 & 5 \\ -9 & 6 & 0 \end{bmatrix} \qquad \ldots\text{(ii)}$$

From (i) and (ii),

$$(\mathbf{AB})^{\mathbf{T}} = \mathbf{B}^{\mathbf{T}}\mathbf{A}^{\mathbf{T}}$$

2.15 SYMMETRIC MATRIX

A square matrix A such that $A' = A$ is called symmetric matrix, i.e., matrix $[a_{ij}]$ is symmetric provided $a_{ij} = a_{ji}$ for all values of i and j.

For example:

$$A = \begin{bmatrix} a & h & g \\ h & b & f \\ g & f & c \end{bmatrix} \text{ is a symmetric matrix.}$$

2.16 SKEW-SYMMETRIC MATRIX

A square matrix A such that $A' = -A$ is called skew-symmetric matrix, i.e., matrix $[a_{ij}]$ is skew-symmetric provided $a_{ij} = -a_{ji}$ for all values of i and j.

For example:

$$A = \begin{bmatrix} 0 & 1 & -3 \\ -1 & 0 & 5 \\ 3 & -5 & 0 \end{bmatrix} \text{ is a skew-symmetric matrix.}$$

Remarks. In a skew-symmetric matrix, we have $a_{ij} = -a_{ji}$. For diagonal elements

$a_{ij} = -a_{ji}$, i.e., $2a_{ij} = 0$, or $a_{ij} = 0$.

Thus, every diagonal element of a skew-symmetric matrix is zero.

Example 16. Show that every matrix can be unequally expressed as the sum of a symmetric and a skew-symmetric matrix.

Solution: Let A be any square matrix.

Now, we have $A = \frac{1}{2}A + \frac{1}{2}A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$ …(i)

Now, $(A + A')' = A' + (A')' = A' + A = A + A'$

(Since matrix addition is commutative)

and $(A - A')' = A' - (A')' = A' - A = -(A - A')$

Hence $(A + A')$ is symmetric and $(A - A')$ is skew-symmetric.

Consequently $\frac{1}{2}(A + A') = P$ (say) is a symmetric matrix and $\frac{1}{2}(A - A') = Q$ (say) is a skew-symmetric matrix.

Hence, $A = P + Q$.

Thus, any square matrix, can be expressed as the sum of symmetric and skew-symmetric matrix.

Uniqueness. To show that this representation is unique, let us suppose that another representation $A = R + S$ is possible, where R is symmetric and S is skew-symmetric, i.e. $R = R'$ and $= S'$.

Now $A' = (R + S)' = R' + S' = R - S$

Also, $A + A' = (R + S) + (R - S) = 2R$

and $A - A' = (R + S) - (R - S) = 2S$

or $R = \frac{1}{2}(A + A')$ and $S = \frac{1}{2}(A - A')$

Hence $A = \frac{1}{2}(A - A') + \frac{1}{2}(A - A')$, is a unique representation.

Example 17. Express $\begin{bmatrix} 2 & 6 & -8 \\ 4 & 2 & 1 \\ -8 & 6 & 13 \end{bmatrix}$ as a sum of a symmetric and skew-symmetric matrix.

Solution: Let $A = \begin{bmatrix} 2 & 6 & -8 \\ 4 & 2 & 1 \\ -8 & 6 & 13 \end{bmatrix}$ $\therefore$ $A' = \begin{bmatrix} 2 & 4 & -8 \\ 6 & 2 & 6 \\ -8 & 1 & 13 \end{bmatrix}$

$\therefore$ $A + A' = \begin{bmatrix} 4 & 10 & -16 \\ 10 & 4 & 7 \\ -16 & 7 & 26 \end{bmatrix}$ $\therefore \frac{1}{2}(A + A') = \frac{1}{2}\begin{bmatrix} 4 & 10 & -16 \\ 10 & 4 & 7 \\ -16 & 7 & 26 \end{bmatrix}$

Which is a symmetric matrix.

Again $\frac{1}{2}(A - A') = \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -5 \\ 0 & 5 & 0 \end{bmatrix}$

Which is a skew-symmetric matrix.

Now $A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$

i.e., $A = \frac{1}{2}\begin{bmatrix} 4 & 10 & -16 \\ 10 & 4 & 7 \\ -16 & 7 & 26 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 2 & 0 \\ -2 & 0 & -5 \\ 0 & 5 & 0 \end{bmatrix}$

Example 18. If A and B are both symmetric then, show that AB is symmetric iff A and B commute.

Solution: Since A and B are symmetric, we have

$$A' = A \text{ and } B' = B$$

Then $(AB)' = B'A'$ (reversal law)

$$= BA = AB, \text{ iff A and B commute}$$

Thus AB is symmetric iff A and B commute.

2.17 ORTHOGONAL MATRIX

A square matrix A is said to be orthogonal if $A'A = AA' = I$

Now, we know that

$$|A'| = |A|$$

Also $|A'A| = |A'||A|$

or $|I| = |A|^2$ or $|A|^2 = 1$

This shows that the matrix A should be non−singular and invertible, if it is orthogonal matrix. Hence the condition

$A'A = I$ Implies that $A^{-1} = A'$

Example 19. Verify that $\mathbf{A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}}$ **is orthogonal.**

Solution: We have $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

Then $A' = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ -2 & 2 & -1 \end{bmatrix}$

We have, $= \frac{1}{9}\begin{bmatrix} 1+4+4 & 2+2-4 & -2+4-2 \\ 2+2-4 & 4+1+4 & -4+2+2 \\ -2+4-2 & -4+2+2 & 4+4+1 \end{bmatrix}$

$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Hence, by definition, matrix A is orthogonal.

CHECK YOUR PROGRESS (C)

1. If A and B are symmetric, then show that $A + B$ is symmetric.

2. If $A = \begin{bmatrix} 5 & 3 \\ 2 & 4 \end{bmatrix}$, verify that $(6A)' = 6A'$.

3. Given: $A = \begin{bmatrix} 2 & 3 \\ 1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & -3 \\ 1 & 4 \end{bmatrix}$ obtain the matrix $2A' + 3B'$.

4. Express $\begin{bmatrix} 2 & 1 & -11 \\ 6 & 13 & 7 \\ -4 & -2 & 1 \end{bmatrix}$ as a sum of a symmetric and skew-symmetric matrix.

Answers

3. $\begin{bmatrix} 10 & 5 \\ -3 & 10 \end{bmatrix}$

4. $\frac{1}{2}\begin{bmatrix} 4 & 7 & -15 \\ 7 & 26 & 5 \\ -15 & 5 & 2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & -5 & -7 \\ 5 & 0 & 9 \\ 7 & -9 & 0 \end{bmatrix}$

2.18 SUM UP

- A Matrix is a particular type of arrangement of m n numbers which are arranged in the form of an ordered set of m rows and n columns.
- There are different types of Matrices like Row, Column, Square, Rectangular, Null, Diagonal, Scalar, Unit, Triangular matrix etc.
- Matrix algebra includes addition of matrices, difference of matrices, multiplication of a matrix by a scalar and matrix multiplication.
- We can obtain the Transpose of a Matrix by interchanging the corresponding rows and columns of a given matrix.

2.19 KEY TERMS

- **Row Matrix:** If a matrix has only one row, it is called a row matrix.
- **Column Matrix:** A matrix consisting of only one column is called a column matrix.
- **Zero or Null Matrix (0):** If every element of an $m \times n$ matrix is zero, the matrix is called a zero matrix.
- **Square Matrix:** Any matrix in which the number of rows are equal to the number of columns is called a Square matrix.
- **Diagonal Matrix:** A square in which all elements except those in the leading diagonal are zero, is called a diagonal matrix.
- **Scalar Matrix:** A diagonal matrix whose diagonal elements are all equal is called a scalar matrix.
- **Identity or Unit Matrix (I):** A scalar matrix in which each of its diagonal elements is unity is called an identity or unit matrix.

- **Trace of a Matrix:** The trace of any square matrix is the sum of its main diagonal elements.
- **Triangular Matrix:** If every element above or below the leading diagonal is zero, the matrix is called a triangular matrix.
- **Transpose of a Matrix**: A matrix obtained by interchanging the corresponding rows and columns of a given matrix.
- **Symmetric Matrix:** A matrix which is equal to its own transpose.
- **Skew Symmetric Matrix:** A matrix which is equal to negative of its own transpose.

2.20 QUESTIONS FOR PRACTICE

Q1.What is matrix? Explain its types.

Q2.What do you mean by equality of matrices?

Q3.Explain addition and difference of matrix.

Q4.What are the properties of Addition matrix?

Q5.What is Multiplication of Matrices? Give its properties.

Q6.Applications of Matrices to Business and Economic Problems

Q7.What do you mean by transpose of a matrix? Give an example

Q8.Explain symmetric matrix and Skew-symmetric matrix.

2.21 FURTHER READINGS

- Mizrahi and John Sullivan. Mathematics for Business and Social Sciences. Wiley and Sons.
- N. D. Vohra, Business Mathematics and Statistics, McGraw Hill Education (India) Pvt Ltd
- J.K. Thukral, Mathematics for Business Studies, Mayur Publications
- J. K. Singh, Business Mathematics, Himalaya Publishing House.

BACHELOR OF ARTS

SEMESTER – VI

QUANTITATIVE METHODS

UNIT 3: DETERMINANTS

STRUCTURE

3.0 OBJECTIVES

After studying the Unit, students will be able to:

- Define the Meaning Determinant
- Calculate the Determinant of a Square Matrix
- Implement the properties of Determinants
- Understand the difference between Minors and Co-Factors
- Find the Inverse of a Matrix
- Solve the Linear Equations with the help of Matrices
- Solve the Linear Equations with help of Determinants

3.1 DETERMINANT

To every square matrix $A = [a_{ij}]_{m \times m}$, we associate a number called determinant of the matrix A. It is denoted by determinant A or $|A|$. The matrix which is not square does not possess determinant.

Finding value of a determinant

1) **Determinant of order 1,** i. e. of 1×1 matrix

   The value of the determinant of order one is the number of which the determinant is formed.

   Thus, $|a_{11}| = a_{11}$

   For example, $|-2| = -2$

2) **Determinant of order 2,** i.e., of 2×2 matrix

   The value of the determinant of order two is found as under:

   $$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = (a_{11}a_{22} - a_{12}a_{21})$$

   For example, $\begin{vmatrix} 2 & 3 \\ 0 & -1 \end{vmatrix}$

   $= (2)(-1) - 3 \times 0$

   $= -2 - 0 = \mathbf{-2}$

3) **Determinant of order 3,** i.e., of 3×3 matrix

   The value of the determinant of order three is found by expanding the determinant by any row or any column.

   For example, $|A| = \Delta = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

   Expanding it by 1st row, we get

   $$\Delta = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

   $$= a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$$

   Similarly, we can expand it by any other row or any column, taking care of the fact the signs meddler for adding the terms will follow the following pattern

   $$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

   Thus, by expanding the determinant by, say, second column, we have

   $$\Delta = -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$= -a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{22}(a_{11}a_{33} - a_{13}a_{31}) - a_{32}(a_{11}a_{23} - a_{13}a_{21})$$

Thus, if we have

$$A = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 5 & 2 \\ 4 & -3 & 1 \end{vmatrix}$$

then

$$|A| = \begin{vmatrix} 2 & -1 & 0 \\ -1 & 5 & 2 \\ 4 & -3 & 1 \end{vmatrix}$$

Expanding it by 'R_1', we get

$$2\begin{vmatrix} 5 & 2 \\ -3 & 1 \end{vmatrix} - (-1)\begin{vmatrix} -1 & 2 \\ 4 & 1 \end{vmatrix} + 0\begin{vmatrix} -1 & 5 \\ 4 & -3 \end{vmatrix}$$

$$= 2(5 + 6) + 1(-1 - 8) + 0(3 - 20)$$

$$= 2(11) + 1(-9) + 0(-17)$$

$$= 22 - 9 + 0 = \mathbf{13}$$

Example 1. Find the value of determinant

$$\begin{vmatrix} 3 & -5 & 8 \\ 6 & -4 & -3 \\ 4 & 2 & 0 \end{vmatrix} \text{by expanding it by}$$

(i) second row and (ii) third column.

Solution: (i) Expanding by the second row,

$$\begin{vmatrix} 3 & -5 & 8 \\ 6 & -4 & -3 \\ 4 & 2 & 0 \end{vmatrix} = -6\begin{vmatrix} -5 & 8 \\ 2 & 0 \end{vmatrix} + (-4)\begin{vmatrix} 3 & 8 \\ 4 & 0 \end{vmatrix} - (-3)\begin{vmatrix} 3 & -5 \\ 4 & 2 \end{vmatrix}$$

$$= -6(-16) - 4(-32) + 3(26)$$

$$= 96 + 128 + 78 = 302$$

(ii) Expanding by the third column,

$$\begin{vmatrix} 3 & -5 & 8 \\ 6 & -4 & -3 \\ 4 & 2 & 0 \end{vmatrix} = 8\begin{vmatrix} 6 & -4 \\ 4 & 2 \end{vmatrix} - (-3)\begin{vmatrix} 3 & -5 \\ 4 & 2 \end{vmatrix} + 0\begin{vmatrix} 3 & -5 \\ 6 & -4 \end{vmatrix}$$

$$= 8(28) - 3(26) + 0$$

$= 224 + 78 = 302$, same as before.

3.2 PROPERTIES OF DETERMINANTS

The following are the important properties of determinants. The students are advised to verify these properties on their own.

(i) If any two rows (or columns) of a determinant are interchanged, the sign of the determinant is changed, the absolute value remaining unaltered, For example,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = -\begin{vmatrix} a_1 & a_3 & a_2 \\ b_1 & b_3 & b_2 \\ c_1 & c_3 & c_2 \end{vmatrix}$$

In this example the second and third columns have been interchanged.

(ii) If every element in any row (or column) of a determinant is multiplied by the same scalar c, the determinant thus obtained is c times the original determinant. Thus,

$$\begin{vmatrix} a_1 & ca_2 & a_3 \\ b_1 & cb_2 & b_3 \\ c_1 & cc_2 & c_3 \end{vmatrix} = c\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

(iii) If determinant rows are changed into columns of columns into rows, the value of the determinant remains unchanged. Thus,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = c\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

(iv) If any two rows (or columns) of a determinant are identical, the value of the determinant is zero. Thus, $\begin{vmatrix} a_1 & b_2 & c_3 \\ a_1 & b_1 & b_1 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0, \begin{vmatrix} 3 & 9 & 5 \\ 2 & 4 & 7 \\ 1 & 3 & 6 \end{vmatrix} = 3\begin{vmatrix} 3 & 3 & 5 \\ 2 & 2 & 7 \\ 1 & 1 & 6 \end{vmatrix} = 3 \times 0 = 0.$

(v) If each element of any row or any column is the sum (or difference) of two quantities, the determinant can be expressed as the sum (or difference) of two determinants of the same order, as given below:

$$\begin{vmatrix} a_1 & a_2 + \alpha & a_3 \\ b_1 & b_2 + \beta & b_3 \\ c_1 & c_2 + \gamma & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & \alpha & a_3 \\ b_1 & \beta & b_3 \\ c_1 & \gamma & c_3 \end{vmatrix}.$$

(vi) If any row (or column) or a multiple thereof is added to or subtracted from any other row (or column), the value of the determinant remains unchanged. Thus,

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 + ka_1 \\ b_1 & b_2 & b_3 + kb_2 \\ c_1 & c_2 & c_3 + kc_2 \end{vmatrix}$$

(vii) The determinant of an identity matrix is always 1"

Consider the determinant of an identity matrix I = $\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}$, |I| = (1)(1) = 1.

Thus, the determinant of any identity matrix is always 1.

(vii) "If any square matrix B with order n×n has a zero row or a zero column, then det(B) = 0"

Consider the determinant of an identity matrix B,

|B| = $\begin{vmatrix} 2 & 2 \\ 0 & 0 \end{vmatrix}$,

|B| = (2)(0) – (2) (0) = 0.

(viii) "If C is upper or a lower-triangular matrix, then det(C) is the product of all its diagonal entries"

Consider an upper triangular matrix C with the diagonal entries 3, 2 and 4. The determinant |C| can be found as: $|\mathbf{C}| = \begin{vmatrix} 3 & 1 & 1 \\ 0 & 2 & 5 \\ 0 & 0 & 4 \end{vmatrix}$

$|\mathbf{C}|$ = 3 *2* 4 = 24

Example 2. Without expanding prove that

$$\begin{vmatrix} 6 & 2 & 7 \\ 15 & 5 & 5 \\ 21 & 7 & 3 \end{vmatrix} = 0$$

Solution: Let $\Delta = \begin{vmatrix} 6 & 2 & 7 \\ 15 & 5 & 5 \\ 21 & 7 & 3 \end{vmatrix}$

Taking '3' common from C_1, we have

$$\Delta = 3\begin{vmatrix} 2 & 2 & 7 \\ 5 & 5 & 5 \\ 7 & 7 & 3 \end{vmatrix}$$

$$= 3(0) = \mathbf{0} \qquad (\because C_1 \text{ and } C_2 \text{ are identical})$$

Example 3. Without expanding evaluate the determinant

$$\begin{vmatrix} \mathbf{42} & \mathbf{2} & \mathbf{5} \\ \mathbf{80} & \mathbf{8} & \mathbf{9} \\ \mathbf{30} & \mathbf{6} & \mathbf{3} \end{vmatrix}$$

Solution: Let

$$\Delta = \begin{vmatrix} 42 & 2 & 5 \\ 80 & 8 & 9 \\ 30 & 6 & 3 \end{vmatrix}$$

Applying $C_1 = C_1 + (-8)C_3$, we get

$$\Delta = \begin{vmatrix} 2 & 2 & 5 \\ 8 & 8 & 9 \\ 6 & 6 & 3 \end{vmatrix} \qquad (\because\ C_1 \text{ and } C_2 \text{ are identical})$$

$$\Rightarrow \qquad \Delta = 0$$

Example 4. Prove that

$$|\mathbf{A}| = \begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{b+c} & \mathbf{a+c} & \mathbf{a+b} \end{vmatrix} = \mathbf{0}$$

Solution:

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ b+c & a+c & a+b \end{vmatrix}$$

Applying $R_1 = R_1 + R_2$, we get

$$|A| = \begin{vmatrix} 1 & 1 & 1 \\ a+b+c & b+a+c & c+a+b \\ b+c & a+c & a+b \end{vmatrix}$$

$$= (a+b+c)\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ b+c & a+c & a+b \end{vmatrix} \quad \{\text{taking } (a+b+c) \text{ common from } R_2$$

$$= (a+b+c) \times (0) \qquad (\because\ R_1 \text{ and } R_2 \text{ are identical})$$

$$= 0$$

Hence proved.

Example 5. Prove that

$$\begin{vmatrix} \mathbf{bc} & \mathbf{a} & \mathbf{a_2} \\ \mathbf{ca} & \mathbf{b} & \mathbf{b_2} \\ \mathbf{ab} & \mathbf{c} & \mathbf{c_2} \end{vmatrix} = \begin{vmatrix} \mathbf{1} & \mathbf{a_2} & \mathbf{a_3} \\ \mathbf{1} & \mathbf{b_2} & \mathbf{b_3} \\ \mathbf{1} & \mathbf{c_2} & \mathbf{c_3} \end{vmatrix}$$

Solution:

$$\text{L.H.S.} = \begin{vmatrix} bc & a & a_2 \\ ca & b & b_2 \\ ab & c & c_2 \end{vmatrix} = \frac{1}{abc}\begin{vmatrix} abc & a_2 & a_3 \\ abc & b_2 & b_3 \\ abc & c_2 & c_3 \end{vmatrix}$$

(by multiplying R_1, R_2, R_3 by abc respectively)

$$= \frac{abc}{abc}\begin{vmatrix} 1 & a_2 & a_3 \\ 1 & b_2 & b_3 \\ 1 & c_2 & c_3 \end{vmatrix} \qquad (\text{Taking abc common from } C_2)$$

$$= \begin{vmatrix} 1 & a_2 & a_3 \\ 1 & b_2 & b_3 \\ 1 & c_2 & c_3 \end{vmatrix} = \text{R.H.S.}$$

Example 6. Without expansion, prove that

$$\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0$$

Solution: Let

$$\Delta = \begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix}$$

$$= (-1)^3 \begin{vmatrix} 0 & -a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} \text{(by taking } (-1) \text{ common from each of 3 rows)}$$

$$= -\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} \qquad \text{(Interchanging rows and columns)}$$

$$= -\Delta$$

$\Rightarrow$ $\Delta = -\Delta$

$\Rightarrow$ $2\Delta = 0$

$\Rightarrow$ $\Delta = 0$

i. e. $\begin{vmatrix} 0 & a & -b \\ -a & 0 & c \\ b & -c & 0 \end{vmatrix} = 0$

Example 7. Prove that

$$\begin{vmatrix} \mathbf{1} & \mathbf{1} & \mathbf{1} \\ \mathbf{a} & \mathbf{b} & \mathbf{c} \\ \mathbf{a^2} & \mathbf{b^2} & \mathbf{c^2} \end{vmatrix} = (\mathbf{a}-\mathbf{b})(\mathbf{b}-\mathbf{c})(\mathbf{c}-\mathbf{a})$$

Solution:

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 0 & 0 & 1 \\ a-b & b-c & c \\ a^2-b^2 & b^2-c^2 & c^2 \end{vmatrix} \quad \text{where } \begin{matrix} C_1 = C_1 - C_2 \\ C_2 = C_2 - C_3 \end{matrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 0 & 0 & 1 \\ 1 & 1 & c \\ a+b & b+c & c^2 \end{vmatrix}$$

$$= (a-b)(b-c)\begin{vmatrix} 1 & 1 \\ a+b & b+c \end{vmatrix}$$

$$= (a-b)(b-c)(b+c-a-b)$$

$$= (a-b)(b-c)(c-a).$$

Example 8. Prove that

$$\begin{vmatrix} \mathbf{-a^2} & \mathbf{ab} & \mathbf{ac} \\ \mathbf{ba} & \mathbf{-b^2} & \mathbf{bc} \\ \mathbf{ac} & \mathbf{bc} & \mathbf{-c^2} \end{vmatrix} = \mathbf{4a^2b^2c^2}$$

Solution:

$$\text{L.H.S.} = abc\begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} \qquad \begin{pmatrix}\text{on taking out a, b and c common} \\ \text{from } R_1, R_2 \text{ and } R_3 \text{ repectively}\end{pmatrix}$$

$$= a^2b^2c^2\begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \qquad \begin{pmatrix}\text{on taking out a, b and c common} \\ \text{from } C_1, C_2 \text{ and } C_3 \text{ repectively}\end{pmatrix}$$

$$= a^2b^2c^2\begin{vmatrix} 0 & 0 & 2 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \qquad (\text{applying } R_1 = R_1 + R_2)$$

$$= a^2b^2c^2 2(1+1) = 4a^2b^2c^2,$$

Example 9. Show that

$$\begin{vmatrix} \mathbf{y+z} & \mathbf{x} & \mathbf{y} \\ \mathbf{z+x} & \mathbf{z} & \mathbf{x} \\ \mathbf{x+y} & \mathbf{y} & \mathbf{z} \end{vmatrix} = (\mathbf{x+y+z})(\mathbf{x-z})^2$$

Solution:

$$\begin{vmatrix} y+z & x & y \\ z+x & z & x \\ x+y & y & z \end{vmatrix} = \begin{vmatrix} 2(x+y+z) & x+y+z & x+y+z \\ z+x & z & x \\ x+y & y & z \end{vmatrix}$$

operating $R_1 = R_1 + R_2 + R_3$

$$= (x+y+z)\begin{vmatrix} 2 & 1 & 1 \\ z+x & z & x \\ x+y & y & z \end{vmatrix}, (\text{taking } (x+y+z) \text{ common from } R_1)$$

$$= (x+y+z)\begin{vmatrix} 0 & 0 & 1 \\ x-z & z-x & x \\ x-y & y-z & z \end{vmatrix}, \qquad \begin{pmatrix}\text{operating } C_1 = C_1 - 2C_2 \\ \text{and } C_2 = C_2 - C_3\end{pmatrix}$$

$$= (x+y+z)\begin{vmatrix} x-z & z-x \\ x-y & y-z \end{vmatrix}$$

$$= (x+y+z)(x-z)\begin{vmatrix} 1 & -1 \\ x-y & y-z \end{vmatrix}, (\text{Taking } (x-z) \text{ common from } R_1)$$

$$= (x+y+z)(x-z)(y-z+x-y)$$

$$= (x+y+z)(x-z)^2.$$

TEST YOUR PROGRESS (A)

1. Evaluate $\begin{vmatrix} 1 & 2 & 5 \\ 2 & 3 & 1 \\ -1 & 1 & 1 \end{vmatrix}$

2. Evaluate $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$

3. Evaluate $\begin{vmatrix} b+c & a+b & a \\ c+a & b+c & b \\ a+b & c+a & c \end{vmatrix}$

4. Show that

(a) $\begin{vmatrix} 1 & a & b+c \\ 1 & b & c+a \\ 1 & c & a+b \end{vmatrix} = 0$ (b) $\begin{vmatrix} 1 & bc & a(b=c) \\ 1 & ca & b(c+a) \\ 1 & ab & c(a+b) \end{vmatrix} = 0$

5. Prove that

$$\begin{vmatrix} 1 & a^2+bc & a^3 \\ 1 & b^2+ca & b^3 \\ 1 & c^2+ab & c^3 \end{vmatrix} = -(b-c)(c-a)(a-b)(a^2+b^2+c^2)$$

6. Show that

$$\begin{vmatrix} a & a^2 & a^3 \\ b & b^2 & b^3 \\ c & c^2 & c^3 \end{vmatrix} = abc(a-b)(b-c)(c-a)$$

7. Prove that (without expanding)

$$\begin{vmatrix} 1+a & b & c \\ a & 1+b & c \\ a & b & 1+c \end{vmatrix} = 1+a+b+c$$

Answers

1. 21, **2.** Xy, **3.** $a^3+b^3+c^3-3abc$

3.3 MINORS

Consider the determinant of a square matrix **A,**

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

When we delete any one row and any one column of $|\mathbf{A}|$, we get a 2×2 determinant. For example, if we strike off the row and column passing through a_{11}, i.e., the first row and first column, we get the determinant as

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

This determinant is called the minor of the element a_{11} in determinant **A.**

Thus, the minor of the element in the determinant of the square matrix may be defined as a determinant which is left after deleting the row and column in which the element lies. The number of minors in a determinant will be equal to the number if elements therein. The following is the list of all nine minors in $|\mathbf{A}|$.

The minors of a_{11}, a_{12} and a_{13} are

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \text{ and } \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \text{ respectively.}$$

The minors of a_{21}, a_{22} and a_{23} are

$$\begin{vmatrix} a_{12} & a_{13} \\ a_{32} & a_{33} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} \text{ and } \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \text{ respectively.}$$

The minors of a_{31}, a_{32} and a_{33} are

$$\begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix} \text{ and } \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \text{ respectively.}$$

In general, the determinant obtains by striking off the ith row and jth column of a matrix $\mathbf{A} = [a_{ij}]_{n \times n}$ is called the minor of a_{ij} in $|\mathbf{A}|$. The minor of element a_{ij} is determinant by $\mathbf{M}_{ij}$.

3.4 CO-FACTORS OF A MATRIX

If we multiply the minor of the element in the ith row and jth column of the determinant of the matrix by $(-1)^{i+j}$ the product is called the co-factor of the element. It is usual to denote the co-factor of an element by the corresponding capital letter. Symbolically

$$\mathbf{A}_{ij} = (-1)^{i+j} \times \text{minor of } a_{ij} \text{ in } \mathbf{A}$$
$$= (-1)^{i+j} \mathbf{M}_{ij}.$$

If we consider a determinant

$$|\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

We get $\mathbf{A}_{11} = (-1)^{1+1} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$,

$$\mathbf{A}_{12} = (-1)^{1+2} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix},$$

$$\mathbf{A}_{23} = (-1)^{2+3} \begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix},$$

$$\mathbf{A}_{31} = (-1)^{3+1} \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}, \text{ and so on.}$$

Note:

1. The sum of the products of the elements of any row (column) of a determinant with the corresponding co-factors is equal to the value of the determinant, i.e.,

$$|\mathbf{A}| = a_{11}\mathbf{A}_{11} + a_{12}\mathbf{A}_{12} + a_{13}\mathbf{A}_{13},$$

$$= a_{12}\mathbf{A}_{12} + a_{22}\mathbf{A}_{22} + a_{32}\mathbf{A}_{32}, \text{ etc.}$$

2. The sum of the products of the elements of any row (column) with the co-factors of the corresponding elements of any other row (column) is zero, i. e.,

$$\mathbf{0} = a_{11}\mathbf{A}_{12} + a_{12}\mathbf{A}_{22} + a_{13}\mathbf{A}_{23},$$

$$= a_{13}\mathbf{A}_{11} + a_{23}\mathbf{A}_{21} + a_{33}\mathbf{A}_{31}, \text{ etc.}$$

Example 10. Find the minors and co-factors of matrix A and use it to evaluate the determinant of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{1} & \mathbf{-4} \\ \mathbf{2} & \mathbf{5} & \mathbf{6} \\ \mathbf{1} & \mathbf{4} & \mathbf{8} \end{bmatrix}$$

Solution: The minors are calculated as follows:

As $\mathbf{M}_{ij} = \text{minor of } a_{ij}$

$\therefore$ $\mathbf{M}_{11} = \begin{vmatrix} 5 & 6 \\ 4 & 8 \end{vmatrix} = 40 - 24 = 16$

$\mathbf{M}_{12} = \begin{vmatrix} 2 & 6 \\ 1 & 8 \end{vmatrix} = 16 - 6 = 10$

$\mathbf{M}_{13} = \begin{vmatrix} 2 & 5 \\ 1 & 4 \end{vmatrix} = 8 - 5 = 3$

$\mathbf{M}_{21} = \begin{vmatrix} 1 & -4 \\ 4 & 8 \end{vmatrix} = 8 + 16 = 24$

$\mathbf{M}_{22} = \begin{vmatrix} 3 & -4 \\ 1 & 8 \end{vmatrix} = 24 + 4 = 28$

$\mathbf{M}_{23} = \begin{vmatrix} 3 & 1 \\ 1 & 4 \end{vmatrix} = 12 - 1 = 11$

$\mathbf{M}_{31} = \begin{vmatrix} 1 & -4 \\ 5 & 6 \end{vmatrix} = 6 + 20 = 26$

$\mathbf{M}_{32} = \begin{vmatrix} 3 & -4 \\ 2 & 6 \end{vmatrix} = 18 + 8 = 26$

$\mathbf{M}_{33} = \begin{vmatrix} 3 & 1 \\ 2 & 5 \end{vmatrix} = 15 - 2 = 13$

Now as co-factor $\mathbf{A}_{ij}$ of element $a_{ij} = (-1)^{i+j}\,\mathbf{M}_{ij}$

$\therefore$ Co-factors

$\mathbf{A}_{11} = (-1)^2\,\mathbf{M}_{11} = +(16) = 16$

$\mathbf{A}_{12} = (-1)^3\,\mathbf{M}_{12} = -(10) = -10$

$\mathbf{A}_{13} = (-1)^4\,\mathbf{M}_{13} = +(3) = 3$

$\mathbf{A}_{21} = (-1)^3\,\mathbf{M}_{21} = -(24) = -24$

$$\mathbf{A}_{22} = (-1)^4\, \mathbf{M}_{22} = +(28) = 28$$
$$\mathbf{A}_{23} = (-1)^5\, \mathbf{M}_{23} = -(11) = -11$$
$$\mathbf{A}_{31} = (-1)^4\, \mathbf{M}_{31} = +(26) = 26$$
$$\mathbf{A}_{32} = (-1)^3\, \mathbf{M}_{32} = -(26) = -26$$
$$\mathbf{A}_{33} = (-1)^6\, \mathbf{M}_{33} = +(13) = 13$$

Also $|\mathbf{A}| = \mathrm{a}_{11}\mathbf{A}_{11} + \mathrm{a}_{12}\mathbf{A}_{12} + \mathrm{a}_{13}\mathbf{A}_{13}$

$$= 3(16) + 1(-10) - 4(3)$$
$$= 48 - 10 - 12 = \mathbf{26.}$$

3.5 ADJOINT OF A MATRIX

Let $\mathbf{A} = [\mathrm{a}_{ij}]$ be a square matrix of order n and let $\mathbf{A}_{ij}$ denote the co-factor of a_{ij} in $|\mathbf{A}|$. Then the adjoint of matrix $\mathbf{A}$ is defined as the transpose of the matrix $[\mathbf{A}_{ij}]$ and is expressed by writing adj $\mathbf{A}$.

Thus, in order to find the adjoint of a matrix, replace each element of this matrix by the co-factor of that element and then transpose this matrix of co-factors. The two operations may also be performed in reverse order, i.e., first find the transpose of the given matrix and then replace each element by its co-factor.

Hence, if
$$\mathbf{A} = \begin{bmatrix} \mathrm{a}_{11} & \mathrm{a}_{12} & \cdots & \mathrm{a}_{1n} \\ \mathrm{a}_{21} & \mathrm{a}_{22} & \cdots & \mathrm{a}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \mathrm{a}_{n1} & \mathrm{a}_{n2} & \cdots & \mathrm{a}_{nn} \end{bmatrix},$$

we have
$$\text{adj } \mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} & \cdots & \mathbf{A}_{1n} \\ \mathbf{A}_{21} & \mathbf{A}_{22} & \cdots & \mathbf{A}_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \mathbf{A}_{n1} & \mathbf{A}_{n2} & \cdots & \mathbf{A}_{nn} \end{bmatrix}$$

where capital letters denote the co-factors of small letters in $|\mathbf{A}|$.

i.e., $\mathbf{A}_{ij}$ = co-factor of a_{ij} in $|\mathbf{A}|$.

Some Properties

(i) If $\mathbf{A}$ is a square matrix,

$$\mathbf{A}.(\text{adj } \mathbf{A}) = (\text{adj } \mathbf{A}).\mathbf{A}$$
$$= |\mathbf{A}|\mathbf{I}$$

Where $\mathbf{I}$ is the matrix of the same order as $\mathbf{A}$.

(ii) If $\mathbf{A} = [a_{ij}]$ is a square matrix of order n, we have

$$|\text{adj } \mathbf{A}| = |\mathbf{A}|^{n-1}, \text{ provided } |\mathbf{A}| \neq 0.$$

(iii) If $\mathbf{A}$ and $\mathbf{B}$ are two square matrices of $n \times n$ order each, we have

$$\text{adj}(\mathbf{AB}) = (\text{adj } \mathbf{B}).(\text{adj } \mathbf{A}).$$

3.6 INVERSE OF A MATRIX

If, for a given square matrix $\mathbf{A}$ of order $\times n$, there exists of matrix $\mathbf{B}$ such that $\mathbf{AB} = \mathbf{BA} = \mathbf{I}_n$ (where $\mathbf{I}_n$ is a unit matrix of order n), the square matrix $\mathbf{B}$ is said to be an inverse of $\mathbf{A}$. We write $\mathbf{B}$ as $\mathbf{A}^{-1}$, read as '$\mathbf{A}$ inverse'. A matrix having an inverse is called an inverse matrix.

From the definition given above, we see that we can talk of an inverse of a square matrix only. Also, we find that, if $\mathbf{B}$ be an inverse of a square matrix only. Also, we find that, if $\mathbf{B}$ be an inverse of **A, A** is also an inverse of **B.**

We already know that

$$\mathbf{A}.(\text{adj } \mathbf{A}) = |\mathbf{A}|\mathbf{I}$$

or $$\mathbf{A}\left(\frac{\text{adj } \mathbf{A}}{|\mathbf{A}|}\right) = \mathbf{I}, \text{ provided } |\mathbf{A}| \neq 0.$$

Hence, $$\mathbf{A}^{-1}\left(\frac{\text{adj } \mathbf{A}}{|\mathbf{A}|}\right), \text{ if } |\mathbf{A}| \neq 0.$$

Thus, we find another from the inverse or reciprocal of the matrix $\mathbf{A}$, which is quite suggestive of the procedure for finding an inverse.

$\mathbf{A}^{-1}$ is said to be the inverse of $\mathbf{A}$ because it possesses the property $\mathbf{AA}^{-1} = \mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$.

Theorem 1. The inverse of a matrix is unique.

Proof. If possible, let $\mathbf{B}$ and $\mathbf{C}$ be two inverses of a square matrix $\mathbf{A}$. Since $\mathbf{B}$ is an inverse of $\mathbf{A}$, we have

$$\mathbf{AB} = \mathbf{BA} = \mathbf{I} \quad \ldots(i)$$

Again, since $\mathbf{C}$ is an inverse of $\mathbf{A}$, we have

$$\mathbf{AC} = \mathbf{CA} = \mathbf{I} \quad \ldots(ii)$$

From (i), we have

$$\mathbf{C}(\mathbf{AB}) = \mathbf{CI} = \mathbf{C} \quad \ldots(iii)$$

Also, from (ii), we have

$$(\mathbf{CA})\mathbf{B} = \mathbf{IB} = \mathbf{B} \quad \ldots(iv)$$

We know that $\mathbf{C}(\mathbf{AB}) = (\mathbf{CA})\mathbf{B}$.

Therefore, from (iii) and (iv) it follows that

$$\mathbf{B} = \mathbf{C}$$

i. e., the inverse is unique.

Theorem 2. A square matrix **A** has an inverse, if and only if $|\mathbf{A}| \neq 0$. OR

A square matrix **A** is invertible, if and only if **A** is non-singular.

Proof. The condition is necessary

Let **B** be the inverse of the matrix **A,** then

$$\mathbf{AB} = \mathbf{I}.$$

Therefore, $|\mathbf{A}||\mathbf{B}| = |\mathbf{I}| = 1$

Hence, $|\mathbf{A}| \neq 0$

The condition is sufficient. Suppose $|\mathbf{A}| \neq 0$. Let us assume that

$$\mathbf{B} = \frac{\text{adj } \mathbf{A}}{|\mathbf{A}|}$$

$$\therefore \quad \mathbf{AB} = \mathbf{A}.\left(\frac{\text{adj } \mathbf{A}}{|\mathbf{A}|}\right)$$

$$= \frac{1}{|\mathbf{A}|}(\mathbf{A}.\text{adj } \mathbf{A}) = \frac{|\mathbf{A}|\mathbf{I}}{|\mathbf{A}|} = \mathbf{I}$$

Similarly, $\mathbf{BA} = \mathbf{I}$

$\therefore \quad \mathbf{AB} = \mathbf{BA} = \mathbf{I}.$

Hence, **A** has an inverse.

Theorem 3. Reversal Law: If **A** and **B** are invertible square matrices of the same order, then **AB** is also invertible and

$$(\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$$

Given A, B are invertible matrices of same order, hence

$$|\mathbf{A}| \neq 0, \quad \text{and} \quad |\mathbf{B}| \neq 0.$$

$\Rightarrow \quad |\mathbf{A}||\mathbf{B}| \neq 0 \quad \Rightarrow \quad |\mathbf{AB}| \neq 0$

$\Rightarrow$ **AB** is invertible.

Let $(\mathbf{AB})^{-1} = \mathbf{C}$, then $(\mathbf{AB})\mathbf{C} = \mathbf{I} = \mathbf{C}(\mathbf{AB})$

Now, $(\mathbf{AB})\mathbf{C} = \mathbf{I} \quad \Rightarrow \quad \mathbf{A}(\mathbf{BC}) = \mathbf{I}$

$\Rightarrow \quad \mathbf{A}^{-1}[\mathbf{A}(\mathbf{BC})] = \mathbf{A}^{-1}$

$\Rightarrow \quad (\mathbf{A}^{-1}\mathbf{A})(\mathbf{BC}) = \mathbf{A}^{-1} \quad \Rightarrow \quad \mathbf{I}(\mathbf{BC}) = \mathbf{A}^{-1}$

$\Rightarrow \quad \mathbf{BC} = \mathbf{A}^{-1} \quad \Rightarrow \quad \mathbf{B}^{-1}(\mathbf{BC}) = \mathbf{B}^{-1}\mathbf{A}^{-1}$

$\Rightarrow \qquad (\mathbf{B}^{-1}\mathbf{B})\mathbf{C} = \mathbf{B}^{-1}\mathbf{A}^{-1} \quad \Rightarrow \qquad \mathbf{IC} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

$\Rightarrow \qquad \mathbf{C} = \mathbf{B}^{-1}\mathbf{A}^{-1}$

Hence, $\qquad (\mathbf{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Example 11. Find the adjoint of the matrix $\mathbf{A} = \begin{bmatrix} -2 & 3 \\ -5 & 4 \end{bmatrix}$

Solution: $\qquad A = \begin{bmatrix} -2 & 3 \\ -5 & 4 \end{bmatrix}$

Here, co-factor of $a_{11} = (-1)^{1+1}\, 4 = 4$

Co-factor of $a_{12} = (-1)^{1+2}\,(-5) = -(-5) = 5$

Co-factor of $a_{21} = (-1)^{2+1}\, 3 = -3$

Co-factor of $a_{22} = (-1)^{2+2}\,(-2) = -2$

Adjoint of $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 4 & 5 \\ -3 & -2 \end{bmatrix}' = \begin{bmatrix} 4 & -3 \\ 5 & -2 \end{bmatrix}$

Example 12. Find adjoint of $\mathbf{A} = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 3 \\ 4 & 1 & 5 \end{bmatrix}$

Solution: $\qquad A = \begin{bmatrix} 2 & 1 & 4 \\ 3 & 1 & 3 \\ 4 & 1 & 5 \end{bmatrix}$

Now, If $\mathbf{A}_{ij}$ = co-factor of a_{ij}

Then Adj $\mathbf{A} = \begin{bmatrix} a_{11} & a_{21} & a_{31} \\ a_{12} & a_{22} & a_{32} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$

$$\text{Adj } \mathbf{A} = \begin{bmatrix} (-1)^{1+1}\begin{vmatrix} 1 & 3 \\ 1 & 5 \end{vmatrix} & (-1)^{1+2}\begin{vmatrix} 3 & 3 \\ 4 & 5 \end{vmatrix} & (-1)^{1+3}\begin{vmatrix} 3 & 1 \\ 4 & 1 \end{vmatrix} \\ (-1)^{2+1}\begin{vmatrix} 1 & 4 \\ 1 & 5 \end{vmatrix} & (-1)^{2+2}\begin{vmatrix} 2 & 4 \\ 4 & 5 \end{vmatrix} & (-1)^{2+3}\begin{vmatrix} 2 & 1 \\ 4 & 1 \end{vmatrix} \\ (-1)^{3+1}\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} & (-1)^{3+2}\begin{vmatrix} 2 & 4 \\ 3 & 3 \end{vmatrix} & (-1)^{3+3}\begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} \end{bmatrix}'$$

$$\text{Adj } \mathbf{A} = \begin{bmatrix} (5-3) & -(15-12) & (3-4) \\ -(5-4) & (10-16) & -(2-4) \\ (3-4) & -(6-12) & (2-3) \end{bmatrix}'$$

$$\text{Adj } \mathbf{A} = \begin{bmatrix} 2 & -3 & -1 \\ -1 & -6 & 2 \\ -1 & 6 & -1 \end{bmatrix}' = \begin{bmatrix} 2 & -1 & -1 \\ -3 & -6 & 6 \\ -1 & 2 & -1 \end{bmatrix}$$

c

Note. The answer can be verified from the fact that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}$, as shown below.

$$\mathbf{A}^{-1}\mathbf{A} = \begin{bmatrix} \frac{1}{3} & \frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{5}{3} & \frac{4}{3} \\ \frac{1}{3} & -\frac{2}{3} & \frac{1}{3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{3}+\frac{8}{3}-1 & \frac{4}{3}-\frac{4}{3} & \frac{2}{3}-\frac{2}{3} \\ -\frac{2}{3}-\frac{10}{3}+4 & -\frac{5}{3}+\frac{8}{3} & -\frac{4}{3}+\frac{4}{3} \\ \frac{1}{3}-\frac{4}{3}+1 & -\frac{2}{3}+\frac{2}{3} & \frac{2}{3}+\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}.$$

Example 14. Calculate the inverse of the matrix

$$\mathbf{A} = \begin{bmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{bmatrix}$$

Solution: Here $|\mathbf{A}| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc.$

Now, $|\mathbf{A}| = 0$, if $ad - bc = 0$.

Let us assume that $ad - bc \neq 0$ so that $\mathbf{A}^{-1}$ exists.

But $\mathbf{A}^{-1} = \frac{\text{adj } \mathbf{A}}{|\mathbf{A}|}$.

Now, we have $A_{11} = d, A_{12} = -c, A_{21} = -b$ and $A_{22} = a$.

∴ $\text{adj } \mathbf{A} = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

And $\mathbf{A}^{-1} = \frac{1}{(ad-bc)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.

Example 15. Find the inverse of the matrix:

$$\begin{bmatrix} 3 & -10 & -1 \\ -2 & 8 & 2 \\ 2 & -4 & -2 \end{bmatrix}$$

Solution: Let $A = \begin{bmatrix} 3 & -10 & -1 \\ -2 & 8 & 2 \\ 2 & -4 & -2 \end{bmatrix}$

$$|\mathbf{A}| = 3(-16+8) + 10(4-4) - 1(8-16)$$

$$= -24 + 0 + 8 = -16 \neq 0 \qquad \therefore\ \mathbf{A}^{-1} \text{ exists}$$

The co-factors are

$\mathbf{A}_{11} = -16 + 8 = -8,$

$\mathbf{A}_{12} = -(4-4) = 0,$

$\mathbf{A}_{13} = 8 - 16 = -8,$

$\mathbf{A}_{21} = -(20-4) = -16,$

$$\mathbf{A}_{22} = -(6+2) = -4,$$

$$\mathbf{A}_{23} = -(-12+20) = -8,$$

$$\mathbf{A}_{31} = -20+8 = -12$$

$$\mathbf{A}_{32} = -(6-2) = -4$$

$$\mathbf{A}_{33} = 24-20 = 4$$

$$\therefore \text{Matrix of co-factors} = \begin{bmatrix} -8 & 0 & -8 \\ -16 & -4 & -8 \\ -12 & -4 & 4 \end{bmatrix}$$

$$\therefore \text{adj } \mathbf{A} = \begin{bmatrix} -8 & -16 & -12 \\ 0 & -4 & -4 \\ -8 & -8 & 4 \end{bmatrix} = -4\begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$$

Then $\mathbf{A}^{-1} = \frac{1}{|\mathbf{A}|}(\text{adj } \mathbf{A}) = \frac{4}{16}\begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 2 & 4 & 3 \\ 0 & 1 & 1 \\ 2 & 2 & -1 \end{bmatrix}$

Example 16. Given the matrix

$$\mathbf{X} = \begin{bmatrix} \mathbf{1} & \mathbf{4} & \mathbf{2} \\ \mathbf{-1} & \mathbf{2} & \mathbf{1} \\ \mathbf{1} & \mathbf{3} & \mathbf{2} \end{bmatrix}, \qquad \textbf{show that} \mathbf{XX^{-1} = I_3}.$$

Solution: We know that $\mathbf{X}^{-1} = \frac{\text{Adj } \mathbf{X}}{|\mathbf{X}|}$

We have $|\mathbf{X}| = 1(4-3) - 4(-2-1) + 2(-3-2) = 3$

Let us find adjoint of **X**

$$A_{11} = (-1)^{1+1}\begin{vmatrix} 2 & 1 \\ 3 & 2 \end{vmatrix} = 1,$$

$$A_{12} = (-1)^{1+2}\begin{vmatrix} -1 & 1 \\ 1 & 2 \end{vmatrix} = 3,$$

$$A_{13} = (-1)^{1+3}\begin{vmatrix} -1 & 2 \\ 2 & 3 \end{vmatrix} = -5,$$

$$A_{21} = (-1)^{2+1}\begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix} = -2,$$

$$A_{22} = (-1)^{2+2}\begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} = 0,$$

$$A_{23} = (-1)^{2+3}\begin{vmatrix} 1 & 4 \\ 1 & 3 \end{vmatrix} = 1,$$

$$A_{31} = (-1)^{3+1}\begin{vmatrix} 4 & 2 \\ 2 & 1 \end{vmatrix} = 0,$$

$$A_{32} = (-1)^{3+2}\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix} = -3,$$

$$A_{33} = (-1)^{3+3}\begin{vmatrix} 1 & 4 \\ -1 & 2 \end{vmatrix} = 6$$

Hence, co-factor matrix = $\begin{bmatrix} 1 & 3 & -5 \\ -2 & 0 & 1 \\ 0 & -3 & 6 \end{bmatrix}$

∴ $\quad \text{Adj } \mathbf{X} = \begin{bmatrix} 1 & -2 & 0 \\ 3 & 0 & -3 \\ -5 & 1 & 6 \end{bmatrix}$

∴ $\quad \mathbf{X}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & -2 & 0 \\ 3 & 0 & -3 \\ -5 & 1 & 6 \end{bmatrix}$

∴ $\quad \mathbf{XX}^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 4 & 2 \\ -1 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 3 & 0 & -3 \\ -5 & 1 & 6 \end{bmatrix}$

$$= \frac{1}{3}\begin{bmatrix} 1\times1+4\times3+2\times-5 & 1\times-2+4\times0+2\times1 & 1\times0+4\times-3+2\times6 \\ -1\times1+2\times3+1\times-5 & -1\times-2+2\times0+1\times1 & -1\times0+2\times-3+1\times6 \\ 1\times1+3\times3+3\times-5 & 1\times-2+3\times0+2\times1 & 1\times0+3\times-3+2\times6 \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathbf{I}_3.$$

Example 17. Show that $\mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$ satisfies the equation $\mathbf{A}^2 - 4\mathbf{A} - 5\mathbf{I} = \mathbf{0}$ where I is identity matrix and 0 denotes the zero matrix. Hence find the inverse of A.

Solution: Here, $\quad \mathbf{A} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

∴ $\quad \mathbf{A}^2 = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \times \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

$$= \begin{bmatrix} 1+4+4 & 2+2+4 & 2+4+2 \\ 2+2+4 & 4+1+4 & 4+2+2 \\ 2+4+2 & 4+2+2 & 4+4+1 \end{bmatrix} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix}$$

Also, $\quad 4\mathbf{A} = \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix}$

and $\quad 5\mathbf{I} = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

∴ $\quad \mathbf{A}^2 - 4\mathbf{A} - 5\mathbf{I} = \begin{bmatrix} 9 & 8 & 8 \\ 8 & 9 & 8 \\ 8 & 8 & 9 \end{bmatrix} - \begin{bmatrix} 4 & 8 & 8 \\ 8 & 4 & 8 \\ 8 & 8 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$

$$= \begin{bmatrix} 9-4-5 & 8-8 & 8-8 \\ 8-8 & 9-4-5 & 8-8 \\ 8-8 & 8-8 & 9-4-5 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0 = \mathbf{R.H.S.}$$

To find $\mathbf{A}^{-1}$.

$$|\mathbf{A}| = \begin{vmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{vmatrix} = 1(1-4) - 2(2-2) + 2(4-2) = 5 \neq 0$$

Hence, $\mathbf{A}^{-1}$ exists.

Multiplying the both sides of the equation

$$\mathbf{A^2 - 4A - 5I} = 0$$

by $\mathbf{A}^{-1}$, we have

$$\mathbf{A^{-1}A^2 - 4A^{-1}A - 5A^{-1}I} = 0$$

or $\qquad \mathbf{A - 4I - 5A^{-1}} = 0$

or $\qquad \mathbf{5A^{-1}} = \mathbf{A} - 4\mathbf{I}$

$$= \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} - \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

$\therefore \qquad \mathbf{A}^{-1} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}.$

TEST YOUR PROGRESS (B)

Q1. For the matrices $\mathbf{A} = \begin{bmatrix} 1 & -2 \\ 3 & 4 \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} 5 & 6 \\ -3 & 2 \end{bmatrix}$ prove that $(\mathrm{AB})^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1}$.

Q2. If $\mathbf{A} = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & -3 \\ 3 & -2 & 4 \end{bmatrix}$, $\mathbf{B}^{-1} = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & -1 \\ 1 & 0 & 2 \end{bmatrix}$, find $(\mathbf{AB})^{-1}$.

Q3. If $\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & -2 & 1 \\ 4 & 2 & 1 \end{bmatrix}$, show that $\mathbf{A}^3 - 23\mathbf{A} - 40\mathbf{I} = 0$, where $\mathbf{I}$ is a unit matrix of order 3 and 0 is null matrix of order 3. Hence (not otherwise), compute A^{-1}.

Q4. Verify $\mathbf{A}^2 + 3\mathbf{A} + 4\mathbf{I} = 0$ for the matrix

$$\mathbf{A} = \begin{bmatrix} -1 & -1 \\ 2 & -2 \end{bmatrix}.$$

Hence, obtain the inverse of **A.**

Q5. Show that the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$$

Satisfies the equation $\mathbf{A}^3 - 6\mathbf{A}^2 + 9\mathbf{A} - 4\mathbf{I} = 0$, and hence, deduce $\mathbf{A}^{-1}$.

Q6. How many minors does a 3×3 matrix have?

Answers

1. adj $\mathbf{A} = \begin{bmatrix} -4 & -3 & -3 \\ 1 & 0 & 1 \\ 4 & 4 & -3 \end{bmatrix}$, $\mathbf{A}^{-1} = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & 3 \end{bmatrix}$;

2. $\frac{1}{19}\begin{bmatrix} 16 & 12 & 1 \\ 21 & 11 & -7 \\ 10 & -2 & 3 \end{bmatrix}$

3. $\mathbf{A}^{-1} = \frac{1}{40}\begin{bmatrix} -4 & 4 & 8 \\ 1 & 11 & 8 \\ 14 & 6 & 8 \end{bmatrix}$,

5. $\frac{1}{4}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$

6. 9

3.7 SOLVING LINEAR EQUATIONS BY MATRIX INVERSE METHOD

With the help of matrices, a set of linear equations can be solved to find unknown values. Suppose we have the following set of linear equations:

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n = b_1$$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n = b_2$$

$$\vdots$$

$$\vdots$$

$$a_{n_1}x_1 + a_{n_2}x_2 + \ldots\ldots + a_{nm}x_n = b_n.$$

These equations may be written in the following way:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n_1} & a_{n_2} & \cdots & a_{nm} \end{bmatrix}; X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}; B = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

If A is a non-singular matrix then

$$X = A^{-1}B, \quad A^{-1} = \frac{\text{Adj. A}}{|A|}$$

or $$X = \frac{\text{Adj. A} \cdot B}{|A|}$$

Criterion of Consistency:

Let $AX = B$ be a system of n-linear equations in n unknowns

Case I: If $|A| \neq 0$, then the system is consistent and has a unique solution given by $X = A^{-1} B$.

Case II: If $|A| = 0$ and $(\text{adj } A)$ **.** $B = 0$, then the system is consistent and has infinite many solutions.

Case III: If $|A| = 0$ and $(\text{adj } A)$ **.** $B \neq 0$, then the system is consistent.

Note: If $|A| = 0$, then system will have infinite numbers of solutions.

Example 18. Solve the system of equations

$$\mathbf{x + y + z = 6}$$

$$\mathbf{x - y + z = 2}$$

$$\mathbf{2x + y - z = 1}$$

Solution: The system of equations in matrix A^{-1} from is $AX = B$, where

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 2 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$$

Now $$A^{-1} = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}$$

∴ $$X = \frac{1}{6}\begin{bmatrix} 0 & 2 & 2 \\ 3 & -3 & 0 \\ 3 & 1 & -2 \end{bmatrix}\begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6 \\ 12 \\ 18 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

∴ $$x = 1, y = 2, z = 3$$

Example 19. Solve the following equations by matrix inverse method

$$\mathbf{x_1 + 2x_2 = 10}$$

$$\mathbf{2x_1 - x_2 = 15}$$

Solution: $A = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}, X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, B = \begin{bmatrix} 10 \\ 15 \end{bmatrix}$

$AX = B$

$|A| = \begin{vmatrix} 1 & 2 \\ 2 & -1 \end{vmatrix} = -1 - 4 = -5$

∴ $|A| = -5$

Co-factor of $1 = -1$,

Co-factor of $2 = -2$

Co-factor of $2 = -2$

Co-factor of $1 = -1$

$\therefore$ Matrix of co-factors $= \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}$

$$\text{Adj.} = \begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{-5}\begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix} \text{ or } A^{-1}B = -\frac{1}{-5}\begin{bmatrix} -1 & -2 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} 10 \\ 15 \end{bmatrix}$$

$$= \frac{-1}{5}\begin{bmatrix} -10 & +(-30) \\ -20 & +(-15) \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -40 \\ -35 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \end{bmatrix}$$

$\therefore$ $\mathbf{x_1 = 8,\ x_2 = 7}$

TEST YOUR PROGRESS (C)

1. Solve the following equations by matrix method:

$$2x - y = 5; \quad 3x + 4y = 7$$

2. Solve the following equations by using matrix method:

$$x + y + z = 6; \quad x + 2y + 3z = 14; \quad x + y + 7z = 24$$

3. Solve the following system of linear equations by using matrix method:

$$4x - 2y = 3; \quad 6x - 3y = 5$$

4. Solve the following equations by matrix method

$$2x_1 + 5x_2 = 11; \quad 4x_1 - 3x_2 = 9$$

5. Solve the following equations by inverse method

$$2x_1 - 2x_2 + 5x_3 = 1; \quad 2x_1 - 4x_2 + 8x_3 = 2; \quad -3x_1 + 6x_2 + 7x_3 = 1$$

6. A salesman has following record of sales during three months for three items A, B and C which have different rates of commission.

Months Sale of Units Total Commission

	A	B	C	Drawn (in Rs.)
January	90	100	120	800
February	130	50	40	900
March	60	100	30	850

Find out rates of commission per item on A, B and C by matrix method.

Answers

1. $\frac{27}{11}, \frac{-1}{11}$ **2.** 1, 2, 3 **3.** Inconsistent

4. $x_1 = 3, x_2 = 1$ 5. $x_1 = \frac{-14}{19}, x_2 = \frac{-7}{38}, x_3 = \frac{4}{19}$ **6.** $x = 2, y = 4, z = 11$

3.8 SOLVING THE EQUATIONS BY CRAMER RULE

It is a method that adopts determinants therefore it is also known as solving equations by the method of determinant. By using this method, in a set of three (say) linear equations, the unknown values x, y and z can be obtained. To solve the equations, 4 determinants are to be computed. They are:

$|D| =$ For the information given in the matrix i. e. determinants of coefficient of x, y, z.

$|D_1| =$ Determinant obtained from $|D|$ by replacing its first column by the values of B.

$|D_2| =$ Determinant obtained from $|D|$ by replacing its second column by the values of B.

$|D_3| =$ Determinant obtained from $|D|$ by replacing its third column by the values of B.

Example 20. Solve the following equations by Cramer's Rule:

$$\mathbf{2x - y = 5}$$

$$\mathbf{x - 4y = -1}$$

Solution: The above equations can be presented in the form of matrices:

$$\begin{vmatrix} 2 & -1 \\ 1 & -4 \end{vmatrix} \begin{vmatrix} x \\ y \end{vmatrix} = \begin{vmatrix} 5 \\ -1 \end{vmatrix}$$

$$AX = B$$

Consider $$|D| = \begin{vmatrix} 2 & -1 \\ 1 & -4 \end{vmatrix} = (-8 + 1) = -7$$

$$|D_1| = \begin{vmatrix} 5 & -1 \\ -1 & -4 \end{vmatrix} = (-20 - 1) = -21$$

$$|D_2| = \begin{vmatrix} 2 & 5 \\ 1 & -1 \end{vmatrix} = -2 - 5 = -7$$

$\therefore$ $$x = \frac{|D_1|}{|D|} = \frac{-21}{-7} = 3,\ y = \frac{|D_2|}{|D|} = \frac{-7}{-7} = 1$$

$\therefore$ $$x = 3,\ y = 1.$$

Example 21. Solve the following equations

$$\mathbf{6x + y - 3z = 5}$$

$$\mathbf{x + 3y - 2z = 5}$$

$$\mathbf{2x + y + 4z = 8}$$

Solution: The above equations can be presented in the form of matrix

$$\begin{bmatrix} 6 & 1 & -3 \\ 1 & 3 & -2 \\ 2 & 1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 8 \end{bmatrix}$$

$$AX = B$$

To solve the equations, consider the determinant

$$|D| = \begin{vmatrix} 6 & 1 & -3 \\ 1 & 3 & -2 \\ 2 & 1 & 4 \end{vmatrix} = 6(12+2) - 1(4+4) - 3(1-6)$$

$$= 6 \times 14 - 8 \times 1 + 5 \times 3 = 84 - 8 + 15 = 91$$

$$|D_1| = \begin{vmatrix} 5 & 1 & -3 \\ 5 & 3 & -2 \\ 8 & 1 & 4 \end{vmatrix} = 5(12+2) - 1(20+16) + 3(5-24)$$

$$= 70 - 36 + 57 = 91$$

$$|D_2| = \begin{vmatrix} 6 & 5 & -3 \\ 1 & 5 & -2 \\ 2 & 8 & 4 \end{vmatrix} = 6(20+16) - 5(4+4) - 3(8-10)$$

$$= 216 - 40 + 6 = 182$$

$$|D_3| = \begin{vmatrix} 6 & 1 & 5 \\ 1 & 3 & 5 \\ 2 & 1 & 8 \end{vmatrix} = 6(24-5) - 1(8-10) + 5(1-6)$$

$$= 114 + 2 - 25 = 91$$

∴ $x = \frac{|D_1|}{|D|} = \frac{91}{91} = 1$

$y = \frac{|D_2|}{|D|} = \frac{182}{91} = 2$

$z = \frac{|D_3|}{|D|} = \frac{91}{91} = 1$

∴ $x = 1, y = 2, z = 1.$

Example 22. The sum three numbers is 20. If we multiply first by 2 and add the second number and subtract the third, we get 23. If we multiply the first by 3 and add second and third to it, we get 46. Find the numbers.

Solution: Let the number be x, y and z repectively. According, the question we have

$x + y + z = 20$ (First Condition)

$2x + y - z = 23$ (Second Condition)

$3x + y + z = 46$ (Third Condition)

The above equations can be presented in the form of a matrix.

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 20 \\ 23 \\ 46 \end{bmatrix}$$

$AX = B$

To solve the equations, consider the determinant

$$|D| = \begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & 1 \end{vmatrix} = 1(1+1) - 1(2+3) + 1(2-3)$$

$$= 2 - 5 - 1 = 2 - 6 = -4$$

$$|D_1| = \begin{vmatrix} 20 & 1 & 1 \\ 23 & 1 & -1 \\ 46 & 1 & 1 \end{vmatrix} = 20(1+1) - 1(23+46) + 1(23-46)$$

$$= 40 - 69 - 23 = 40 - 92 = -52$$

$$|D_2| = \begin{vmatrix} 1 & 20 & 1 \\ 2 & 23 & -1 \\ 3 & 46 & 1 \end{vmatrix} = 1(23+46) - 20(2+3) + 1(92-69)$$

$$= 69 - 100 + 23 = 92 - 100 = -8$$

$$|D_3| = \begin{vmatrix} 1 & 1 & 20 \\ 2 & 1 & 23 \\ 3 & 1 & 46 \end{vmatrix} = 1(46-23) - 1(92-69) + 20(2-3)$$

$$= 23 - 23 - 20 = 0 - 20 = -20$$

$\therefore$ $$x = \frac{|D_1|}{|D|} = \frac{-52}{-4} = 13$$

$$y = \frac{|D_2|}{|D|} = \frac{-8}{-4} = 2$$

$$z = \frac{|D_3|}{|D|} = \frac{-20}{-4} = 5$$

Thus, $x = 13$, $y = 2$ and $z = 5$.

Example 23. Solve the following system of equations

$$\mathbf{x + 3y + 4z = 8}$$

$$\mathbf{2x + y + 2z = 5}$$

$$\mathbf{5x + 2z = 7}$$

Solution: $$A = \begin{vmatrix} 1 & 3 & 4 \\ 2 & 1 & 2 \\ 5 & 0 & 2 \end{vmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} 8 \\ 5 \\ 7 \end{bmatrix}$$

$$AX = B$$

$$|D| = 1(2-0) - 3(4-10) + 4(0-5) = 2 + 18 - 20 = 0$$

Since $|D| = 0$, the system may have infinite number of solutions

Now solve any two of the given equations. Solving the last two equations i.e. equation (ii) and (iii), we get

$$2x + y + 2k = 5, 5x + 2k = 7$$

$\Rightarrow \qquad x = \frac{7-2k}{5}$

Putting these values of x in equation (ii) we get

$$y = 5 - 2x - 2k = 5 - 2\left(\frac{7-2k}{5}\right) - 2k$$

$\Rightarrow \qquad y = \frac{11-6k}{5}$

Thus, all the infinite solutions of the given system are given by

$$x = \frac{7-2k}{5}, y = \frac{11-6k}{5}, z = k$$

Where k is arbitrary real number. By putting different values of k, we can get different solutions. For example, putting $k = 1$, we get a particular solution $x = 1, y = 1, z = 1$.

TEST YOUR PROGRESS (D)

1. Solve the following equations using Cramer's rule:

$$2x - y = 1; \quad 7x - 2y = -7$$

2. By Cramer's rule, solve the following equations:

$$2x - y + z = 4; \qquad x + 3y + z = 12; \qquad 3x + 2y + z = 5$$

3. Solve the following equations by Cramer's rule:

$$x + y + z = 9; \qquad 2x + 6y + 7z = 55; \qquad 2x + y - z = 0$$

4. Solve the following equations by Cramer's rule

$$2x - y + 3z = 9$$
$$x + 3y - z = 4$$
$$3x + 2y + z = 10$$

5. Solve the following system of equations

(a) $5x - 6y + 4z = 15; \qquad 7x + 4y - 3z = 49; \quad 2x + y + 6z = 46$

(b) $x + y + z = 9; \quad 2x + 6y + 7z = 55; \quad 2x + y - z = 0$

6. Using determinants, show that the following system of equations has infinite number of solutions.

$$x + 3y + 4z = 8; \qquad 2x + y + 2z = 5; \qquad 5x + 2z = 7$$

7. Solve the following set of equations

$$x + y + z = 1$$
$$x + 2y + 3z = 4$$
$$x + 3y + 5z = 7$$

8. A manufacturer is manufacturing two types of products A and B. L_1 and L_2 are two machines which are used for manufacturing these two types of products. The time taken by both A and B on machines is given below:

	Machine L_1	Machine L_2
Product A	20 hrs.	10 hrs.
Product B	10 hrs.	20 hrs.

If 60 hours is the time available on each machine. Calculate the number of units of each type manufactured using matrix method.

Hint: $20x + 10y = 600$

$10x + 20y = 600$

Answers

1. $x = -3, y = -7$

2. $x = -\frac{20}{7}, y = \frac{9}{7}, z = 11$

3. $x = 1, y = 3, z = 5$

4. $x = 1, y = 2, z = 3$

5. **(a)** $x = 3, y = 4, z = 6;$ **(b)** $x = 1, y = 3, z = 5$

6. For $k = 1, y = 1, z = 1$

7. Infinite solutions, $x = k - 2, y = 3 - 2k, z = k$

8. $x = 20, y = 20.$

3.9 SUM UP

- To every square matrix, we associate a single number that gives us an important information about that matrix. That number is called determinant of the matrix .
- The matrix which is not square does not possess determinant.
- Determinant is used to know whether the matrix is invertible or not.
- Determinant of any square matrix can be easily calculated by applying its various properties.
- Matrices and Determinants can be used to solve a system of linear equations.
- There are two methods to solve linear equations i.e. Method of Determinants (Crammer's Rule) and Matrix Inverse Method.

3.10 KEY TERMS

- **Determinant:** To every square matrix, we associate a single number that gives us an important information about that matrix. That number is called determinant of the matrix .
- **Minor:** The minor of the element in the determinant of the square matrix may be defined as a determinant which is left after deleting the row and column in which the element lies.
- **Adjoint of a Matrix:** A matrix obtained by interchanging the rows and columns of the co-factor matrix of the given square matrix.
- **Inverse of a Matrix**: The inverse of any square matrix 'A' is another square matrix 'B' such that AB = BA = I.

3.11 QUESTIONS FOR PRACTICE

Q1. Show that

$$\begin{vmatrix} a-b-c & 2b & 2c \\ 2a & b-c-a & 2c \\ 2a & 2b & c-a-b \end{vmatrix} = (a+b+c)^3$$

Q2. Prove that

$$\begin{vmatrix} a^2 & bc & ac+c^2 \\ a^2+ab & b^2 & ac \\ ab & b^2+bc & c^2 \end{vmatrix} = 4a^2b^2c^2$$

Q3. Show that

$$\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc\left(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)$$

Q4. If $\mathbf{A} = \begin{bmatrix} 4 & 3 & 3 \\ -1 & 0 & -1 \\ -4 & -4 & -3 \end{bmatrix}$,

Find adj **A**, and hence, find the inverse of **A.**

Q5. Verify the theorem $\mathbf{A}.(\text{adj } \mathbf{A}) = (\text{adj } \mathbf{A}).\mathbf{A} = |\mathbf{A}|\mathbf{I}_3$ for the matrix,

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 4 & 3 \end{bmatrix}.$$

Q6. If $\mathbf{A} = \begin{bmatrix} 3 & 4 & 6 \\ 1 & 2 & 3 \\ 8 & 5 & 10 \end{bmatrix}$, verify that $\mathbf{A}^{-1}\mathbf{A} = \mathbf{I}.$

Q7. What are the Properties of Determinants?

Q8. $\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (a-b)(b-c)(c-a)$

Q9. Define Co-Factors of a Matrix.

Q10. Explain Adjoint of a Matrix.

Q11. What is Inverse of a Matrix?

Q12. How to Solving Equations by Cramer Rule?

3.12 FURTHER READINGS

- Mizrahi and John Sullivan. Mathematics for Business and Social Sciences. Wiley and Sons.
- N. D. Vohra, Business Mathematics and Statistics, McGraw Hill Education (India) Pvt Ltd
- J.K. Thukral, Mathematics for Business Studies, Mayur Publications
- J. K. Singh, Business Mathematics, Himalaya Publishing House.

BACHELOR OF ARTS

SEMESTER – VI

QUANTITATIVE METHODS

UNIT 4: DATA AND METHODS: TYPES OF DATA; METHOD OF DATA COLLECTION

STRUCTURE

4.0 OBJECTIVES

After completing this unit, the learner should be able to know:

- Meaning of data

- describe their types of data
- define primary data
- know the different methods of collection of primary data
- define secondary data
- to get familiar with the different sources of collection of secondary data

4.1 MEANING OF DATA

In statistics, "data" refers to the raw information that is collected, either through observation, measurement, or experimentation. It's the building block of any statistical analysis. In social sciences, data are stated as values or facts, together with their accompanying study design, code books, research reports, etc., and are used by researchers for secondary analysis. At one extreme, economics and demography have been heavily quantitative materials or observations. Sociology and, more recently, political science, fall between these two extremes. The change in research orientation on the subject can be seen with changing data, especially with data relating to public opinion.

4.2 TYPES OF DATA

As in sciences, data in social sciences are also organised into different types so that their nature can be easily understood. The following categorization is normally observed in social sciences:

4.2.1 Data about continuity

Data regarding continuity can be categorised as follows:

a) Continuous data: Continuous data are an infinite set of possible values. Between a range, there are infinite possible values. For example, the height of an individual is not restricted to values like 155 cm. and after that to 156 cm. It can be 155.59 cm. or 155.99 cm. - continuous value.

b) Discrete data: The discrete data are finite or potentially countable sets of values. For example, the number of members in a library. It can be 2,575. or 2,599 but certainly not 2,599Y2. Similarly, the number of citizens in a country. the number of vehicles registered is an example of discrete data.

4.2.2 Data with reference to time

There are two types of data under this category. These are:

a) Time series data: Data recorded in chronological order across time are referred to as time series data. It takes different values at different times, e.g., the number of books added to a library in different years, the monthly production of steel in a plant, yearly intake of students in a university.

b) Cross-sectional data: This refers to data for the same unit or for different units at a point in time, e.g., data across sections of people, regions or segments of the society.

4.2.3 Data concerning a number of characteristics

Data can also be categorised based on several variables considered. These are:

a) Univariate data: Univariate data are obtained when one characteristic is used for observation, e.g., the performance of a student in a given class.

b) Bivariate data: Bivariate data results when instead of one, two characteristics are measured simultaneously, e.g., height and weight of tenth class students.

c) Multivariate data: Multivariate data consists of observations on three or more characteristics, e.g., family size, income, and savings in a metropolitan city in India.

4.2.4 Data about characteristic

Data can be categorised based on the characteristics as follows:

a) Qualitative data: it is also known as categorical data, describes the data that fits into the categories. Qualitative data are not numerical. The categorical information involves variables that describe the features such as a person's gender, hometown, etc. Categorical measures are defined in terms of natural language specifications, but not in terms of numbers. Sometimes categorical data can hold numerical values (quantitative value), but those values do not have mathematical sense. For example, birthdate, favorite sport, and school postcode. Here, the birthdate and school postcode hold the quantitative value, but it does not give numerical meaning. It can be further classified as nominal and ordinal data.

b) Quantitative Data: Quantitative data is numerical and can be mathematically computed. Quantitative data measure uses different scales, which can be classified as nominal scale, ordinal scale, interval scale and ratio scale. Often (not always), such data includes measurements of something. Quantitative approaches address the 'what' of the program. They use a systematic standardized approach and employ methods such as surveys and ask questions. Quantitative approaches have the advantage that they are cheaper to implement, are standardized so comparisons can be easily made and the effect size can usually be measured.

Mixed Data: Mixed methods approach as design, combining both qualitative and quantitative research data, techniques and methods within a single research framework. Mixed methods approaches may mean many things, i.e., some different types of methods in a study or at different points within a study or using a mixture of qualitative and quantitative methods. Mixed methods encompass multifaceted approaches that combine to capitalize on strengths and reduce weaknesses that stem from using a single research design.

4.2.5 Data regarding origin

Data under this category can be put as follows:

a) Primary data: The data obtained first-hand from individuals by direct observation, counting, and measurement or by interviews or mailing a questionnaire are called primary data. It may be complete enumeration or sampling, e.g., data collected from a market survey.

b) Secondary data: The data collected initially for the purpose and already published in books or reports but used later on for some other purposes are referred to as secondary data. For example, data collected from census reports, books, data monographs, etc.

CHECK YOUR PROGRESS (A)

Q1. What do you mean by data?

Ans.____

____

Q2. What are the types of data available in statistics?

Ans.____

____

4.3 DATA COLLECTION

The most important part of conducting business research is data collection. To fulfill research objectives and respond to research questions, it involves gathering relevant data and insights. Data collection refers to the systematic process of gathering information and data to address research objectives or answer specific research questions. It involves collecting relevant facts, figures, observations, or responses from various sources or participants, using specific methods or instruments. The purpose of data collection is to obtain correct, consistent, and meaningful data that can be analyzed as well as interpreted to draw insights and make informed decisions. It is a critical step in the research process as the quality of the data collected directly impacts the validity and reliability of the research findings. Several important factors for data collection in business research are Identify Data Sources, Determine Sample Size and Selection, Design Data Collection Instruments, Pilot Testing, Data Collection Procedures, Ensure Data Quality, Record and Organize Data, and Data Verification and Cleaning.

There are mainly two types of data namely Primary Data and Secondary Data. Both have their own importance.

Primary Data: Data that are collected by an investigator agency or institution for a specific purpose and these people are first to use these data, are called primary data. Primary data are original data that are collected for the first time for a specific purpose. Such data are published by authorities who themselves are responsible for their collection. For example, population census data collected by the Government in a country is primary data for that Government.

Secondary Data: The data obtained/gathered by an investigator agency or institution from a source that already exists, are called secondary data. Secondary data may be available in the form of published or unpublished sources. The data becomes secondary for those researchers who use it later. In case you have decided to collect primary data for your investigation, you have to identify the sources from where you can collect that data.

Broadly speaking we can divide the sources of secondary data into two categories: published sources and unpublished sources.

A) Published Sources

- Official publications of the government at all levels- Central, State, Union
- Official publications of international bodies like IMF, UNESCO, WHO, etc.
- Newspapers and Journals of repute, both local and international.
- Reports submitted by reputed economists, research scholars, universities, and commissions of inquiry, if made public.
- Central Statistical Organisation (C.S.0.), National Sample Survey Organisation (N.S.S.O.), Reserve Bank of India Publications (R.B.I.), Labour Bureau.

B) Un-published Sources

- Unpublished findings of certain inquiry committees.
- Research workers' findings.

- Unpublished material found with Trade Associations, Labour Organisations and Chambers of Commerce.

4.4 METHODS OF PRIMARY DATA COLLECTION

After the investigator is convinced that the gain from primary data outweighs the money cost, effort and time, she/he can go in for this. She/he can use any of the following methods to collect primary data:

a. Direct Personal Investigation
b. Indirect Oral Investigation
c. Use of Local Reports/ agencies to get information
d. Questionnaire Method
e. Schedules sent through enumerators

a) Direct Personal Investigation: Here the investigator collects information personally from the respondents. She/ he meets them personally to collect information. This method requires much from the investigator such as:

- She/he should be polite, unbiased and thoughtful.
- She/he should know the local conditions, customs and traditions
- She/he should be intelligent possessing good observation power.
- She/he should use simple, easy, and meaningful questions to extract information.

This method is suitable only for intensive investigations. It is a costly method in terms of money, effort and time. Further, the personal bias of the investigator cannot be ruled out and it can do a lot of harm to the investigation. The method is a complete flop if the investigator does not possess the above-mentioned qualities.

b) Indirect Oral Investigation: Method This method is generally used when the respondents are reluctant to part with the information due to various reasons. Here, the information is collected from a witness or from a third party who is directly or indirectly related to the problem and possesses sufficient knowledge. The person(s) who is/are selected as informants must possess the following qualities:

- They should possess full knowledge of the issue.
- They must be willing to reveal it faithfully and honestly.

- They should not be biased and prejudiced.
- They must be capable of expressing themselves to the true spirit of the inquiry.

c) Use of Local Reports: This method involves the use of local newspapers, magazines, and journals by the investigators. The information is collected by local press correspondents and not by the investigators. Needless to say, this method does not yield sufficient and reliable data The method is less costly but should not be adopted where a high degree of accuracy or precision is required.

d) Questionnaire Method: It is the most important and systematic method of collecting primary data, especially when the inquiry is quite extensive. This method entails creating a questionnaire (a collection of questions in the research area with a chance for respondents to fill in their replies) and mailing it to the respondents with a deadline for responding quickly. the respondents are asked to extend their full cooperation by providing accurate responses and timely submission of the completed questionnaire. By assuring them that the information they provided in the questionnaire will be kept secure and hidden, respondents are also given a sense of security. The investigator typically pays the return postal costs by mailing a self-addressed, stamped envelope to achieve a speedy and better response. Researchers, individuals, non-governmental organisations, and occasionally even the government are involved in this technique.

In other words, this method of data collection is quite popular, particularly in case of big inquiries. The questionnaire is mailed to respondents who are expected to read and understand the questions and write down their replies in the space meant for the questionnaire itself. The respondents have to answer the questions on their own. A Questionnaire is sent to persons with a request to answer the questions and return the questionnaire Questions are printed in a definite order, and mailed to samples who are expected to read the questions understand the questions, and write the answers in the provided space.

e) Schedules sent through Enumerators: Using enumerators for primary data collection is a common practice in various research studies, surveys, and data collection efforts. Enumerators are individuals responsible for collecting data directly from respondents in the field. this is the method of obtaining answers to the questions in a form that is filled out by the interviewers or enumerators (the field agents who put these questions) in a face-to-face situation with the respondents. A questionnaire is a list of questions that the respondent himself answers in his own handwriting. Schedules sent through the enumerators' is the major data collecting technique that is commonly used. This is the case because the earlier ways that have been explained thus far have some drawbacks that this method does not. With the schedule (a list of questions), the enumerators directly contact the respondents, ask them questions, and record their responses.

The questionnaire in primary data is divided into two parts:

1) General introductory part which contains questions regarding the identity of the respondent and contains information such as name, address, telephone number, qualification, profession, etc.
2) Main question part containing questions connected with the inquiry. These questions differ from inquiry-to-inquiry Preparation of the questionnaire is a highly specialized job and is perfected with experience. Therefore, some experienced persons should be associated with it.

THE MAIN ASPECTS OF THE QUESTIONNAIRE

The main aspects of the questionnaire general form structured questionnaire alternatives or yes or no type questions are asked Easy to interpret the data but unuseful for the survey which is aimed to probe for attitudes, and reasons for certain actions Unstructured Questionnaire open-ended questions

- Respondents give answers in their own words based on the pre-test researcher can decide about which type of questionnaire should be used Question Sequence. It should be clear and smoothly moving (the relation of one question to another should be readily apparent First question is important for creating interest in respondent's mind.
- Questions which give stress memory or personal character and wealth should be avoided as opening questions Easier questions should be at the start of the questionnaire General to specific questions should be the sequence of questions Question Formulation and Wording Questions should be easily understood Questions should be simple and concrete.
- Closed questions are easy to handle but this is like fixing the answers in people's mouths. So, depending upon the problem for which the survey is going on both close-ended and open-ended questions may be asked in the Questionnaire. Words having ambiguous meanings should be avoided, catchwords, words with emotional connotations, and dangerous words should be avoided

- Essentials of Good Questionnaire: It should Short & simple Questions should arranged in logical sequence (From Easy to difficult) Technical terms should avoided Some control questions which indicate the reliability of the respondent (To Know consumption first expenditure and then weight or qty of that material) • Questions affecting the sentiments of the respondents should avoid Adequate space for answers should be provided in questionnaire Provision for uncertainty (do not know, No preference) Directions regarding the filling of the questionnaire should be given Physical Appearance - - Quality of paper, color

DRAFTING AND FRAMING A QUESTIONNAIRE

Drafting and framing a questionnaire is a critical step in primary data collection. A well-designed questionnaire ensures that you gather relevant and reliable data to address your research objectives. The following few important points should be kept in mind while drafting a questionnaire:

(i) Clearly outline the research objectives and the specific information you want to collect through the questionnaire. Identify the key research questions that need to be answered.

(ii) Make sure your questions are easy to understand. Avoid nonsense and complex language. Keep sentences and questions short and to the point.

(iii) The task of soliciting information from people in the desired form and with sufficient accuracy is the most difficult problem. By their nature people are not willing to reveal any information because of certain fears. Many times, they provide incomplete and faulty information. Therefore, the respondents must be taken into confidence. They should be assured that their individual information will be kept confidential and no part of it will be revealed to tax and other government investigative agencies. This is very essential indeed. Where providing information is not legally binding, the informant has to be sure and convinced that the results of the survey will help the authorities to frame policies that will ultimately benefit them. Some element of good salesmanship is also required in the investigation.

(iv) Decide the questions that will be included in the questionnaire. Typical sorts of queries include:

- Closed-ended inquiries: Those who respond select from a set of predetermined responses (such as multiple-choice inquiries).

- Open-ended inquiries: Those that respond provide their own, individual responses.
- Questions using a Likert scale: Determine if respondents agree or disagree with a statement using a scale (such as 1 to 5).
- Semantic differential questions: Request a rating on a scale of good to bad or satisfied to dissatisfied from respondents.

(v) Questions hurting the sentiments of respondents should not be asked. These include questions on his gambling habits, sex habits, indebtedness, etc.

(vi) Questions involving lengthy and complex calculations should be avoided because they require tedious extra work in which the respondent may lack both interests as well as capabilities.

4.5 LIMITATIONS OF PRIMARY DATA COLLECTION

Primary data refers to data collected firsthand through direct observation, surveys, interviews, experiments, or other data collection methods. While primary data can be valuable for research and analysis, it also has certain limitations. Here are some common limitations of primary data:

1. **Cost and time**: Collecting primary data can be a time-consuming and costly process. It requires resources to design research instruments, recruit participants, conduct data collection, and analyze the data. Therefore, primary data collection may be impractical or unaffordable.
2. **Limited sample size:** Primary data collection often involves a smaller sample size compared to secondary data sources. The sample size may be constrained by factors such as time, budget, or accessibility of the target population. A small sample size may limit the generalizability of the findings to a larger population.
3. **Sampling bias**: Similar to the limitations of statistics, primary data collection can be susceptible to sampling bias. If the sample is not representative of the population of interest, the findings may not accurately reflect the characteristics or behaviors of the larger population. Careful attention must be given to sampling methods to minimize bias.
4. **Response bias**: Response bias occurs when participants in a study provide inaccurate or misleading responses. It can be influenced by factors such as social desirability bias (participants providing responses they think are socially acceptable) or recall bias

(participants inaccurately remembering past events). Response bias can undermine the validity and reliability of primary data.

5. **Subjectivity and researcher bias**: Primary data collection methods often involve interaction between the researcher and participants. The subjective interpretation and biases of the researcher can unintentionally influence the data collection process and the responses obtained. Researchers need to be aware of their own biases and take steps to minimize their impact on the data.
6. **Limited scope**: Primary data collection typically focuses on specific research questions or objectives. While this targeted approach can yield detailed insights into specific areas of interest, it may not capture a broader range of factors or provide a comprehensive understanding of the phenomenon being studied. Using secondary data or employing a mixed-methods approach can help overcome this limitation.
7. **Ethical considerations:** Primary data collection involves ethical considerations regarding participant privacy, informed consent, and data protection. Researchers must adhere to ethical guidelines and obtain necessary approvals, which can introduce additional time and logistical constraints.

Understanding these limitations of primary data can help researchers and analysts make informed decisions about data collection methods and consider the strengths and weaknesses of primary data concerning their research objectives. It may also be beneficial to supplement primary data with secondary data sources to enhance the breadth and depth of the analysis.

CHECK YOUR PROGRESS (B)

Q1. What do you mean by primary data?

Ans.____

____

Q2. Define the Mailed questionnaire method and schedules sent through enumerators.

Ans.____

____

Q3. Give limitations of primary data.

Ans.____

____

4.6 METHODS OF SECONDARY DATA COLLECTION

As direct investigation, though desirable, is costly in terms of money, time and efforts. Alternatively, information can also be obtained through a secondary source. It means drawing or collecting data from the already collected data of some other agency. Technically, the data so collected are called secondary data.

Secondary data sources in statistics refer to existing data that has been collected by someone else or for a different purpose but can be utilized for statistical analysis. These sources provide a wealth of information that can be used to explore research questions, test hypotheses, and derive insights. Here are some common secondary data sources used in statistics:

1. **Government agencies**: Government agencies at the local, national, and international levels collect and maintain a vast amount of statistical data. Examples include census data, labor statistics, economic indicators, crime rates, health statistics, and demographic information. These datasets are often publicly available and can provide valuable insights into various social, economic, and demographic trends.

2. **Research organizations and institutes**: Many research organizations and institutes conduct surveys, studies, and data collection efforts for specific research purposes. These organizations may focus on topics such as education, public health, social issues, or specific industries. Their datasets can provide detailed information on specific domains or research areas.

3. **International organizations**: International organizations, such as the World Bank, International Monetary Fund (IMF), United Nations (UN), and World Health Organization (WHO), collect and maintain extensive datasets on global development, economics, health, and social indicators. These datasets cover a wide range of countries and can be used for comparative analysis and cross-country studies.

4. **Academic institutions**: Universities and research institutions often conduct research studies and surveys, resulting in datasets that can be valuable for statistical analysis. These datasets may cover various disciplines, including social sciences, psychology, economics, education, and more. Academic institutions often make their datasets available to researchers, subject to certain restrictions and ethical considerations.

5. **Nonprofit organizations**: Nonprofit organizations focused on specific causes or social issues often collect data related to their mission. These organizations may conduct surveys, compile reports, or collaborate with other entities to collect data. Their datasets can provide insights into areas such as poverty, environmental issues, human rights, and social justice.

6. **Commercial data providers**: There are commercial entities that collect, aggregate, and sell datasets on various industries, market trends, consumer behavior, and more. These datasets can be useful for market research, business analytics, and understanding consumer preferences and trends.

7. **Online platforms and social media:** Online platforms and social media networks generate vast amounts of data. This data includes user-generated content, interactions, behaviors, and demographic information. While accessing and analyzing this data may require specific permissions and compliance with privacy regulations, it can offer insights into online behavior, sentiment analysis, and social network analysis.

When using secondary data sources, researchers should consider factors such as the data quality, reliability, representativeness, and potential limitations or biases. It is essential to critically evaluate the data source and ensure that it aligns with the research objectives and analytical requirements

4.7 LIMITATIONS OF SECONDARY DATA

Although the secondary source is cheap in terms of money, time and effort, utmost care should be taken in their use. It is desirable that such data should be vast and reliable and the terms and definitions must match the terms and definitions of the current inquiry. The suitability of the data may be judged by comparing the nature and scope of the present inquiry with that of the original inquiry. Secondary data will be reliable if these were collected by unbiased, intelligent and trained investigators. The time period to which these data belong should also be properly scrutinized.

Secondary data refers to data that is collected by someone else for a different purpose but can be utilized for research or analysis. While secondary data can be convenient and cost-effective, it also has certain limitations. Here are some common limitations of secondary data collection:

1. **Lack of control over data collection:** Since secondary data is collected by others,

researchers have no control over the data collection process. This can result in data that may not perfectly align with the research objectives or may lack specific variables or measures that the researcher requires. The data may not have been collected with the same level of rigor or precision as desired.

2. **Data relevance and accuracy**: The relevance and accuracy of secondary data can vary. It may be challenging to find secondary data that precisely matches the research needs, as the data may be outdated or collected using different methodologies. In some cases, the data may contain errors, inconsistencies, or missing values, which can affect its reliability and validity.

3. **Limited contextual information**: Secondary data may lack detailed information about the context in which it was collected. Understanding the specific circumstances, conditions, or nuances surrounding the data collection process may be crucial for accurate interpretation and analysis. Without sufficient contextual information, the researcher may face challenges in fully understanding and interpreting the data.

4. **Potential bias and validity concerns**: Secondary data may contain inherent biases or limitations introduced by the original data collection process. The biases could be due to the research design, sampling methods, or data collection instruments used. Researchers must critically evaluate the reliability and validity of the secondary data source to ensure its suitability for their research objectives.

5. **Incompatibility and inconsistency**: When working with secondary data from multiple sources, researchers may encounter issues of incompatibility and inconsistency. The data may have been collected using different formats, classifications, or units of measurement, making it challenging to combine or compare the data effectively. Harmonization or standardization efforts may be necessary to address these issues.

6. **Limited control over variables**: Secondary data may not include all the variables of interest to the researcher. Certain variables that are critical for the research objectives may be missing, limiting the scope of analysis or preventing the investigation of specific relationships or factors.

7. **Data availability and access**: Accessing certain types of secondary data can be challenging due to restrictions, copyright issues, or proprietary considerations. Researchers may face

limitations in obtaining the specific data they need or may need to rely on aggregated or summarized data, which may not provide the level of detail required for the research.

Despite these limitations, secondary data can still be a valuable resource for researchers, providing a foundation for analysis, hypothesis generation, and comparison with primary data. Researchers should critically evaluate the quality and relevance of the secondary data and consider its limitations in the interpretation and analysis process.

4.8 PRECAUTIONS TO COLLECT SECONDARY DATA

According to Prof. A.L. Bowley, "It is never safe to take the published statistics at their face value without knowing their meaning and limitations and it is always necessary to criticize the arguments that can be based upon them."

Collecting secondary data involves using information that has already been collected and published by someone else. It is crucial to take precautions to ensure the reliability and validity of the data. Here are some precautions to consider when collecting secondary data:

- Source Credibility: Ensure that the source of the secondary data is reputable, reliable, and unbiased. Academic journals, government publications, and well-established organizations are generally trustworthy sources.
- Date and Currency: Check the publication date of the secondary data. Using outdated information may lead to inaccuracies. Always aim for the most recent and relevant data.
- Purpose of Data Collection: Understand the original purpose for which the data was collected. Make sure it aligns with your research objectives. Misinterpretation of the data's intended use can lead to flawed conclusions.
- Sampling Techniques: Evaluate the sampling methods employed in the original data collection. Understanding the sampling strategy helps assess the generalizability and representativeness of the data.
- Data Accuracy and Reliability: Cross-verify data from multiple sources to ensure accuracy and reliability. Inconsistencies or contradictions between sources may indicate potential issues with the data.
- Documentation and Metadata: Examine the documentation and metadata associated with the

secondary data. Understand the variables, units of measurement, and any transformations applied to the data.

- Data Collection Methods: Be aware of the methods used in the original data collection process. Different methods may introduce biases or limitations that need to be considered in your analysis.
- Data Limitations: Recognize and acknowledge any limitations associated with the secondary data. This may include missing variables, incomplete datasets, or potential biases.
- Ethical Considerations: Respect copyright and intellectual property rights when using secondary data. Ensure that you have the legal right to access and use the data for your research.

By carefully considering these precautions, researchers can enhance the reliability and validity of the secondary data they use in their studies.

In other words, by using secondary data, we should take special note of the following factors.

1) Reliable, 2) Suitable, and 3) Adequate.

1) Reliable: Firstly, the reliability of data has to be the obvious requirement of any data, and more so of secondary data. The user must make himself/herself sure about it. For this, he must check whether data were collected by reliable, trained and unbiased investigators from dependable sources or not.

2) Suitable: Secondly, we should see whether data belong to almost the same type of class of people or not. To look at and compare the given inquiry's objectives, nature, and scope with the original research. To verify that all of the terms and units were uniformly defined throughout the previous investigation and that these definitions are appropriate for the current investigation as well. For instance, a unit can be defined in multiple ways depending on its context, such as a household, wage, price, farm, etc. The secondary data will be considered inappropriate for the present research if the units were identified differently in the original investigation than what we want. lastly, consider the variations in data collecting periods and consistency of conditions comparing the original investigation and the present investigation.

3) Adequate: Third, even if the secondary data are reliable and suitable, it might not be adequate for the particular inquiry's objectives. This happens if the original data refers to an area or a period that is much larger or smaller than the needed one, or when the coverage given in the initial research was too narrow or too wide than what is desired in the current research. Therefore, it is make sure that due to the gap of time, the conditions prevailing then are not much different from the conditions of today in respect of habits, customs, fashion, etc. Of course, we cannot hope to get exactly the same conditions.

Secondary data should not only be reliable and suitable, but also adequate for the present inquiry. It is always desirable that the available data be much more than required by the inquiry. For example, data on, say, consumption pattern of a state cannot be derived from the data on its major cities and towns.

CHECK YOUR PROGRESS (C)

Q1. Explain the method to collect secondary data.

Ans.____

____

Q2. Give two limitations of secondary data.

Ans.____

____

4.8 SUM UP

Data Collection Methods Data / Statistics are quantitative information and can be distinguished as sample or census data; primary or secondary data. We require information for an investigation that can be gathered from either a primary source or a secondary source. Both require statistical surveys, which have two stages: planning and execution. The investigator should choose the primary or secondary sources, census or sample inquiry, type of statistical units and measurement units, level of precision desired, and other factors during the design stage. In the execution stage, the chief investigator has to set up administration, select and train field staff and supervise the entire process of data collection. Using secondary data from published or unpublished sources requires caution because they can lead to several problems. The questionnaire method is the most crucial of all survey methods.

4.9 QUESTIONS FOR PRACTICE

A. Short Answer Type Questions

Q1. What is data?

Q2. Define collection of data.

Q3. Define Primary data

Q4. Define Secondary data

Q5. What is a sample?

Q6. Define quantitative data

Q7. Explain qualitative data

Q8. List out the methods of Primary data.

Q9. List out the methods of Secondary data.

B. Long Answer Type Questions

Q1.Explain the types of data in detail.

Q2.What are the techniques for the collection of data?

Q3.What are the sources of primary data?

Q4.What is a questioner? What are the points to keep in mind before drafting questioner?

Q5.Explain the term secondary data with its sources.

Q6.Define primary data, explain its limitations.

Q7.What is secondary data? Also explain its limitations.

Q8.What are the precautions to collecting secondary Data?

4.10 SUGGESTED READINGS

- Abebe, J. Daniels, J.W. Mckean, "Statistics and Data Analysis".
- Clarke, G.M. & Cooke, D., "A Basic course in Statistics", Arnold.
- David M. Lane, "Introduction to Statistics".
- S.C. Gupta and V.K. Kapoor, "Fundamentals of Mathematical Statistics", SultanChand & Sons, New Delhi.

BACHELOR OF ARTS

SEMESTER – VI

QUANTITATIVE METHODS

UNIT 5: MEASURES OF CENTRAL TENDENCY: MEAN, MEDIAN, MODE

STRUCTURE

5.0 OBJECTIVES

After studying the Unit, students will be able to:

- Meaning of Averages.
- Features of good measure of Average.
- Find different types of Averages for various types of data.
- Understand the relation that exists between different types of Averages.

5.1 INTRODUCTION

We can say that the modern age is the age of Statistics. There is no field in modern life in which statistics is not used. Whether it is Business, Economics or Education. In government Planning or any other field of our life, statistics is used everywhere. Business managers use statistics for business decision-making, Economists use statistics for economic planning, Investors use statistics for future forecasting and so on. Many techniques in statistics help us for all these purposes. Average or Central Tendency is one such technique that is widely used in statistics. This technique is used almost in every walk of life.

5.2 MEANING OF AVERAGE OR CENTRAL TENDENCY

Average or Central tendency is the most used tool of statistics. This is the tool without which statistics is incomplete. In simple words, we can say that the Average is the single value that is capable of representing its series. It is the value around which other values in the series move. We can define Average as the single typical value of the series which represents the whole data of the series. "An average is a single value within the range of data that is used to represent all values in the series. Since an average is somewhere within the range of the data, it is also called a measure of Central Value" **- Croxton and Cowden**

5.3 OBJECTIVES AND FUNCTIONS OF AVERAGE

1. **Single Value Representing Whole Data:** In statistics data can be shown with the help of tables and diagrams. But sometimes the data is very large and it is not easy to present in a table or graph. So, we want to represent that data in summarised form. Average helps us to represent data in summarised form. For example, the data on the national income of India is very large but when we calculate per capita income it gives us an idea of the national income.
2. **To Help in Comparison**: In case we want to compare two different series of data, it is very difficult to compare. There are many difficulties like many items in the series may be different. In such cases, the average helps us in making the comparison. For example, if we want to compare the income of people living in different countries like India and Pakistan, we can do so by calculating per capita income which is a form of average.
3. **Conclude the Universe from Sample:** This is one of the important functions of average. If we take the average of a sample, we can draw certain conclusions about the universe from such an Average. For example, mean of a sample is representative of its universe.
4. **Base on other Statistical Methods:** Many Statistical Techniques are based on average. If we don't know the average, we cannot apply those techniques. For example, Dispersion, Skewness, Index Number are based on average.
5. **Base of Decision Making**: Whenever we have to make a certain decision, average plays a very crucial role in the decision making. From the average we could have an idea about the data and based on that information, we can make a decision. For example, a company can decide on its sales based on average yearly sales of the past few years.
6. **Precise Relationship**: Average helps us to find out if there is a precise relation between two variables or two items. It also removes the biases of the person making the analysis. For example, if you say that Rajesh is more intelligent than Ravi it is only our observation and does not make any precise relation. If we compare the average marks of both the students, we could have a precise relation.

7. **Helpful in Policy Formulation**: The average helps the government formulate the policy. Whenever the government has to formulate economic policy it considers various averages like per capita income, average growth rate, etc.

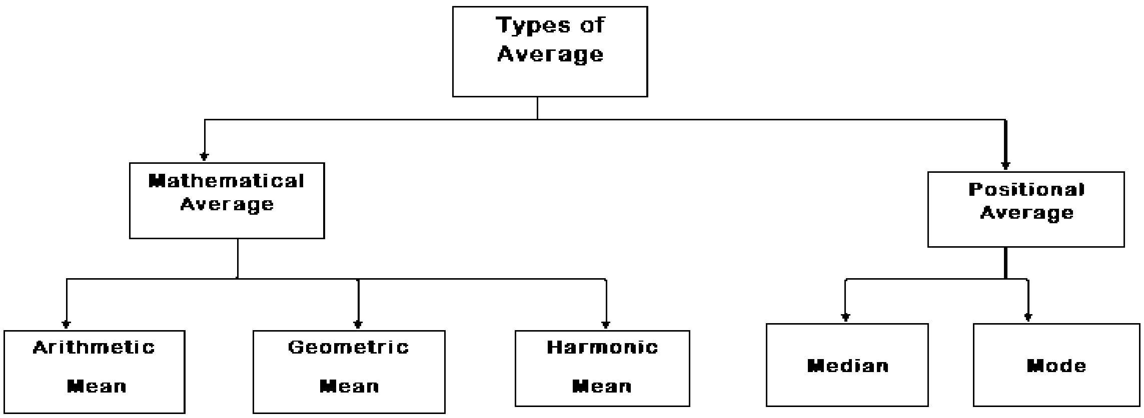
5.4 REQUISITE / FEATURES OF GOOD AVERAGE

1. **Rigidly Defined:** A good measure of average is having a clear-cut definition and no confusion in the mind of a person who is calculating the average. If a person applies his discretion while calculating the average, we cannot say that the average is a good measure.
2. **Easy to Compute**: Good averages do not involve much calculation and are easy to compute. A good average can be calculated even by a person having less knowledge of Statistics. If it is very difficult to calculate the average, we cannot regard it as a good measure.
3. **Based on all Observations:** A good average must consider all the values or data that is available in the series. If the average is based on only a few observations of the series, we cannot say that it is a good measure of the average.
4. **Not affected by Extreme Values**: A good measure of average is not affected by the extreme values present in the Data. Sometimes data contains values that are not within normal limits, these values are called extreme values. If average is affected by these extreme values, we cannot claim that the average is a good measure.
5. **Representative of whole Series:** A good measure of average represents characteristics of a whole series of data.
6. **Easy to Understand:** A good measure of average is not only easy to understand but also easy to interpret.
7. **Not Affected by Fluctuations in the Sampling:** If we take one sample from the universe and calculate average, then we draw another sample from the same universe and calculate the average again, there must not be much difference between these two averages. If the average significantly changes with the change in the sample, we cannot treat it as a good measure of average.

5.5 MEASURES OF CENTRAL TENDENCY

There are many methods through which we can calculate average or central tendency. We can divide these methods into two categories that are Algebraic Method and the Positional Average. Algebraic methods are those in which the value of average depends upon the mathematical formula used in the average. The mathematical average can further be divided into three categories that are

Arithmetic Mean, Geometric Mean and Harmonic Mean. On the other hand, positional averages are those averages that are not based on the mathematical formula used in the calculation of average rather these depend upon the position of the variable in the series. As these depend upon the position of the variable, these averages are not affected by the extreme values in the data. The following chart shows different types of averages.



5.6 ARITHMETIC MEAN

It is the most popular and most common measure of average. It is so popular that for a common man the two terms Arithmetic Mean and Average are the same thing. However, in reality, these two terms are not same and the arithmetic mean is just one measure of the average. We can define the arithmetic mean as:

"The value obtained by dividing the sum of observations by the number of observations".

So arithmetic mean is very easy to calculate, what we have to do is just add up the value of all the items given in the data and then we have to divide that total by the number of items in the data. The arithmetic mean is represented by a symbol A. M. or $\overline{X}$.

5.6.1 Arithmetic Mean in Case of Individual Series

Individual series are those series in which all the items of the data are listed individually. There are two methods of finding arithmetic mean in the individual series. These two methods are the Direct method and Shortcut Method.

1. **Direct Method** According to this method calculation of mean is very simple and as discussed above, we have to just add the items and then divide it by number of items. Following are the steps in the calculation of mean by the direct method:
   1. Suppose our various items of the data are X_1, X_2, X_3 ………………….. X_n

2. Add all the values of the series and find $\sum X$.
3. Find out the number of items in the series denoted by n.
4. Calculate the arithmetic mean by dividing sum value of observation by the number of observations using the following formula:

$$\overline{X} = \frac{\mathbf{X1 + X2 + X3 + --------Xn}}{\mathbf{N}} = \frac{\sum \mathbf{X}}{\mathbf{N}}$$

Where $\overline{X}$= Mean, N = Number of items, $\sum X$ = Sum of observation

Example 1. The daily income of 10 families is as given below (in rupees):

130, 141, 147, 154, 123, 134, 137, 151, 153, 147

Find the arithmetic mean by direct method.

Solution: Computation of Arithmetic Mean

Serial No.	Daily Income (in Rs.) X
1	130
2	141
3	147
4	154
5	123
6	134
7	137
8	151
9	153
10	147
N = 10	$\sum X = 1417$

$$\text{A. M.}, \overline{X} = \frac{X1 + X2 + X3 + --------XN}{N} = \frac{\sum X}{N} = \frac{1417}{10} = \text{Rs. } 141.7$$

2. **Short Cut Method:** Normally this method is used when the value of items is very large and it is difficult to make calculations. Under this method, we take one value as mean which is known as assumed mean and deviations are calculated from this as you mean. This method is also known as is assumed mean method. Following are the steps of this method:
   1. Suppose our various items of the data are X_1, X_2, X_3 ………………….. X_n
   2. Take any value as assumed mean represented by 'A'. This value may be any value among data or any other value even if that is not presented in data.

3. Find out deviations of items from assumed mean. For that deduct an Assumed value from each value of the data. These deviations are represented as 'dx'
4. Find the sum of the deviations represented by $\sum dx$.
5. Find out the number of items in the series denoted by n.
6. Calculate the arithmetic mean dividing sum deviations of the observation with the number of observations using the following formula:

$$\overline{X} = \mathbf{A} + \frac{\sum \mathbf{dx}}{\mathbf{N}}$$

Where $\overline{X}$= Mean, A = Assumed Mean, N = Number of items, $\sum dx$ = Sum of deviations

Example 2. Calculate A. M. by short - cut method for the following data

R. No.	1	2	3	4	5	6	7	8	9	10
Marks	50	60	65	88	68	70	83	45	53	58

Solution: Let assumed Mean (A) be 60

R. No.	Marks (X)	dx = X − A
1	50	-10
2	60	0
3	65	5
4	88	28
5	68	8
6	70	10
7	83	23
8	45	-15
9	53	-7
10	58	-2
N=10		$\sum dx$=40

As $\overline{X} = A + \frac{\sum dx}{N}$

$\Rightarrow$ $\overline{X} = 60 + \frac{40}{10} = 60 + 4$

$\Rightarrow$ $\overline{X} = 64$ Marks

5.6.2 Arithmetic Mean in case of Discrete Series

In individual series if any value is repeated that is shown repeatedly in the series. It makes series lengthy and makes calculation difficult. In case of discrete series, instead of repeatedly showing the items we just group those items and the number of times that item is repeated is shown as frequency. In the case of discrete series, we can calculate Arithmetic mean. By using Direct Method and Shortcut Method.

1. Direct Method: In indirect method we multiply the value of items (X) with their respective frequency (f) to find out the product item (fX). Then we take up sun of the product and divide it with the number of items. Following are the steps

1. Multiply the value of items (X) with their respective frequency (f) to find out the the product item (fX)
2. Add up the product so calculated to find ∑fX.
3. Find out the number of items in the series denoted by n.
4. Calculate arithmetic mean dividing sum of the product by the number of observations using following formula:

$$\overline{X} = = \frac{\sum \mathbf{fX}}{\mathbf{N}}$$

Where $\overline{X}$= Mean, N = Number of items, ∑fX = Sum of product of observations.

Example 3. Find the average income

Daily Income (in rupees)	200	500	600	750	800
No. of Workers	2	1	4	2	1

Solution:

Daily Income (Rs.) X	No. of Workers	Daily Income (Rs.) X
200	2	400
500	1	500
600	4	2400
750	2	1500
800	1	800
	$\sum f = 10$	$\sum fX = 5600$

∴ Average Income $\overline{X} = \frac{\sum fX}{\sum f} = \frac{5600}{10} = \text{Rs.} 560$

2. **Short Cut Method:** Under this method, we take one value as mean which is known as assumed mean and deviations are calculated from this as you mean. Then average is calculated using assumed mean. Following are the steps of this method:

1. Suppose our items of the data are 'X' and its corresponding frequency is 'f'.
2. Take any value as assumed mean represented by 'A'.

3. Find out deviations of items from assumed mean. For that deduct an Assumed value from each value of the data. These deviations are represented as 'dx'
4. Multiply the values of dx with the corresponding frequency to find out product denoted by fdx
5. Find sum of the product so calculated represented by ∑fdx.
6. Find out the number of items in the series denoted by n.
7. Calculate arithmetic mean dividing sum deviations of the observation with the number of observations using following formula:

$$\overline{X} = \mathbf{A} + \frac{\sum \mathbf{fdx}}{\mathbf{N}}$$

Where $\overline{X}$= Mean, A = Assumed Mean, N = Number of items

∑fdx = Sum of product of deviation with frequency.

Example 4. From the following data find out the mean height of the students.

Height (in cms.)	154	155	156	157	158	159	160	161	162	163
No. of Students	1	6	10	22	21	17	14	5	3	1

Solution: Let the Assumed Mean (A) be 150

Height in cms. X	No. of students f	$dX = (X - A) = X - 150$	fdX
154	1	4	4
155	6	5	30
156	10	6	60
157	22	7	154
158	21	8	168
159	17	9	153
160	14	10	140
161	5	11	55
162	3	12	36
163	1	13	13
	$\sum f = 100$		$\sum fdX = 813$

Applying the formula

$$\overline{X} = A + \frac{\sum fdX}{\sum f}$$

We get $\overline{X} = 150 + \frac{813}{100}$

$$= 150 + 8.13 \qquad = 158.13$$

∴ Mean Height = 158.13 cm

5.6.3 Arithmetic Mean in Case of Continuous Series

Continuous series is also known as Grouped Frequency Series. Under this series the values of the observation are grouped into various classes with some upper and lower limits. For example, classes like 10-20, 20-30, 30-40, and so on. In classes 10-20 lower limit is 10 and upper limit is 20. So, all the observations having values between 10 and 20 are put in this class interval. A similar procedure is adopted for all class intervals. The procedure of calculating Arithmetic Mean is a continuous series just like a discrete series except that instead of taking values of observations we take mid value of the class interval. The mid value is represented by 'm' and is calculated using following formula:

$$\mathbf{m} = \frac{\textbf{Lower Limit} + \textbf{Upper Limit}}{\mathbf{2}}$$

1. **Direct Method:** In indirect method, we multiply the mid values (m) with their respective frequency (f) to find out the product item (fm). Then we take up sun of the product and divide it by the number of items. Following are the steps
   1. Multiply the mid values (m) with their respective frequency (f) to find out the the product item (fm)
   2. Add up the product so calculated to find $\sum$fm.
   3. Find out the number of items in the series denoted by n.
   4. Calculate arithmetic mean by dividing sum of the product with the number of observations using following formula:

$$\overline{x} = = \frac{\sum \mathbf{fm}}{\mathbf{N}}$$

Where $\overline{x}$= Mean, N = Number of items, $\sum$fm = Sum of product of observations of mean and frequencies.

Example 5. Calculate the arithmetic mean of the following data:

Class Intervals (C.I.)	100-200	200-300	300-400	400-500	500-600	600-700
f	4	7	16	20	15	8

Solution:

Class Intervals C.I.	Mid Value m	Frequency f	fm
100-200	150	4	600
200-300	250	7	1750

300-400	350	16	5600
400-500	450	20	9000
500-600	550	15	8250
600-700	650	8	5200
		$\sum f = 70$	$\sum fm = 30{,}400$

As $\overline{x} = \frac{\sum fm}{\sum f}$

∴ $\overline{x} = \frac{30{,}400}{70}$ $= 434.3$

2. **Short Cut Method:** This method of mean is almost similar to calculation in the discrete series but here the assumed mean is selected and then the deviation are taken from mid value of the observations. Following are the steps of this method:

1. Calculate the Mid Values of the series represented by 'm'.
2. Take any value as assumed mean represented by 'A'.
3. Find out deviations of items from assumed mean. For that deduct Assumed value from mid values of the data. These deviations are representing as 'dm'
4. Multiply the values of dm with corresponding frequency to find out product denoted by fdm
5. Find sum of the product so calculated represented by ∑fdm.
6. Find out the number of items in the series denoted by n.
7. Calculate arithmetic mean dividing sum deviations of the observation with the number of observations using following formula:

$$\overline{x} = \mathbf{A} + \frac{\sum \mathbf{fdm}}{\mathbf{N}}$$

Where $\overline{x}$= Mean, A = Assumed Mean, N = Number of items

∑fdm = Sum of product of deviation from mid values with frequency.

Example 6. Calculate the mean from the following data

Daily Wages (Rs.)	0-100	100-200	200-300	300-400	400-500	500-600	600-700	700-800	800-900
No. of Workers	1	4	10	22	30	35	10	7	1

Solution: Let the assumed mean, $A = 150$

Daily Wages (Rs.) C.I.	No. of Workers f	Mid Value m	dm = m − A (m − 150)	fdm

0-100	1	50	-100	-100
100-200	4	150	0	0
200-300	10	250	100	1000
300-400	22	350	200	4400
400-500	30	450	300	9000
500-600	35	550	400	14,000
600-700	10	650	500	5000
700-800	7	750	600	4200
800-900	1	850	700	700
	$\sum f = 120$			$\sum fdm = 38{,}200$

As $\overline{X} = A + \frac{\sum fdm}{\sum f}$

$= 150 + \frac{38{,}200}{120}$

$= 150 + 318.33 = 468.33$

$\Rightarrow \overline{X} = 468.33$

3. **Step Deviation Method:** Step Deviation method is the most frequently used method of finding Arithmetic Mean in case of continuous series. This method is normally used when the class interval of the various classes is same. This method makes the process of calculation simple. Following are the steps of this method:

   1. Calculate the Mid Values of the series represented by 'm'.
   2. Take any value as assumed mean represented by 'A'.
   3. Find out deviations of items from assumed mean. For that deduct Assumed value from mid values of the data. These deviations are representing as 'dm'.
   4. Find out if all the values are divisible by some common factor 'C' and divide all the deviations with such common factor to find out dm' which is dm/c
   5. Multiply the values of dm' with corresponding frequency to find out product denoted by fdm'
   6. Find sum of the product so calculated represented by ∑fdm'.
   7. Find out the number of items in the series denoted by n.
   8. Calculate arithmetic mean dividing sum deviations of the observation with the number of observations using following formula:

$$\overline{X} = \mathbf{A} + \frac{\sum \mathbf{fdm'}}{\sum \mathbf{f}} \times \mathbf{C}$$

Where $\overline{X}$= Mean, A = Assumed Mean, N = Number of items, C = Common Factor

$\sum$fdm' = Sum of product of deviation after dividing with common factors and multiplying it with frequency.

Example 7. Use step deviation method to find $\overline{X}$ for the data given below:

Income (Rs.)	1000-2000	2000-3000	3000-4000	4000-5000	5000-6000	6000-7000
No. of Persons	4	7	16	20	15	8

Solution: Let the assumed mean A = 4500

Income (Rs.) C.I.	No of Persons f	Mid Value m	$dm = m - A = (m - 4500)$	$dm' = \frac{dm}{C}$ $C = 1000$	fdm'
1000-2000	4	1500	-3000	-3	-12
2000-3000	7	2500	-2000	-2	-14
3000-4000	16	3500	-1000	-1	-16
4000-5000	20	4500	0	0	0
5000-6000	15	5500	1000	1	15
6000-7000	8	6500	2000	2	16
	$\sum f = 70$				$\sum fdm' = -11$

As $\overline{X} = A + \frac{\sum fdm'}{\sum f} \times C$

∴ $\overline{X} = 4500 + \frac{(-11)}{70} \times 1000$

$= 4500 - \frac{1100}{7}$ $= 4500 - 157.14$

$= 4342.86$ $\overline{X} = 4342.86$

Other Special case of Continuous Series

5.6.4 Arithmetic Mean in case of Cumulative Frequency Series:

The normal continuous series give frequency of the particular class. However, in case of cumulative frequency series, it does not give frequency of particular class rather it gives the total of frequency including the frequency of preceding classes. Cumulative frequency series may be of two types, that are 'less than' type and 'more than' type. For calculating Arithmetic mean in cumulative frequency series, we convert such series into the normal frequency series and then apply the same method as in case of normal series.

Less than Cumulative Frequency Distribution

Example 8. Find the mean for the following frequency distribution:

Marks Less Than	10	20	30	40	50	60
No. of Students	5	15	40	70	90	100

Solution: Convert the given data into exclusive series:

Marks C. I.	No. of Students f	Mid Value m	$dm = m - A$ $A = 25$	$dm' = \frac{dm}{C}$ $C = 10$	fdm'
0-10	5	5	-20	-2	-10
10-20	15-5=10	15	-10	-1	-10
20-30	40-15=25	25	0	0	0
30-40	70-40=30	35	10	1	30
40-50	90-70=20	45	20	2	40
50-60	100-90=10	55	30	3	30
	$\sum f = 100$				$\sum fdm' = 80$

As $\overline{x} = A + \frac{\sum fdm'}{\sum f} \times C$

$\Rightarrow$ $\overline{x} = 25 + \frac{80}{100} \times 10 = 33$

$\Rightarrow$ $\overline{x} = 33$

More Than Cumulative Frequency Distribution

Example 9. Find the mean for the following frequency distribution

Marks More Than	0	10	20	30	40	50	60	70	80	90
No. of Students	80	77	72	65	55	43	28	16	10	8

Solution: Convert the given data into exclusive series

Marks C. I.	No. of Students f	Mid Value	$dm = m - A$ $A = 55$	$dm' = \frac{dm}{C}$ $C = 10$	fdm'
0-10	80-77=3	5	-50	-5	-15
10-20	77-72=5	15	-40	-4	-20
20-30	72-65=7	25	-30	-3	-21
30-40	65-55=10	35	-20	-2	-20
40-50	55-43=12	45	-10	-1	-12
50-60	43-28=15	55	0	0	0
60-70	28-16=12	65	10	1	12
70-80	16-10=6	75	20	2	12
80-90	10-8=2	85	30	3	6
90-100	8	95	40	4	32

	$\sum f = 80$				$\sum fdm' = -26$

As $\quad \overline{X} = A + \frac{\sum fdm'}{\sum f} \times C$

$\therefore \quad \overline{X} = 55 + \frac{(-26)}{80} \times 10 \quad = 55 - \frac{13}{4} \quad = \frac{220-13}{4}$

$= \frac{207}{4} = 51.75 \Rightarrow \quad \overline{X} = 51.75$

5.6.5 Arithmetic Mean in Case of Unequal Class Interval Series:

Sometimes the class interval between two classes is not the same, for example, 10-20, 20-40 etc. These series are known as unequal class interval series. However, it does not affect the finding of arithmetic mean as there is no precondition of equal class interval in the case of arithmetic mean. So, the mean will be calculated in usual manner.

Example 10. Calculate Mean of the data is given below:

C.I.	4-8	8-20	20-28	28-44	44-68	68-80
f	3	8	12	21	10	6

Solution:

C. I.	f	Mid Value m	$dm = m - A$ $A = 26$	fdm
4-8	3	6	-20	-60
8-20	8	14	-12	-96
20-28	12	24	-2	-24
28-44	21	36	+10	210
44-68	10	56	+30	300
68-80	6	74	+48	288
	$\sum f = 60$			$\sum fdm = 618$

As $\quad \overline{X} = A + \frac{\sum fdm}{\sum f}$

$\Rightarrow \quad \overline{X} = 26 + \frac{618}{60} \quad = 26 + 10.3 = 36.3$

$\Rightarrow \quad \overline{X} = 36.3$

5.6.6 Combined Arithmetic Mean:

Sometimes we have the knowledge of mean of two or more series separately but we are interested in finding the mean that will be obtained by taking all these series as one series, such mean is called combined mean. It can be calculated using the following formula.

$$\overline{\mathbf{X}_{12}} = \frac{\mathbf{N_1}\overline{\mathbf{X_1}} + \mathbf{N_2}\overline{\mathbf{X_2}}}{\mathbf{N_1} + \mathbf{N_2}}$$

Where N_1 = Number of items in first series. N_2 = Number of of items in second series

$\overline{X_1}$ = Mean of first series, and $\overline{X_2}$ = Mean of second series

Example 11. Find the combined mean for the following data

	Firm A	Firm B
No. of Wage Workers	586	648
Average Monthly Wage (Rs.)	52.5	47.5

Solution: Combined mean wage of all the workers in the two firms will be

$$\overline{X_{12}} = \frac{N_1\overline{X_1}+N_2\overline{X_2}}{N_1+N_2}$$

Where N_1 = Number of workers in Firm A

N_2 = Number of workers in Firm B

$\overline{X_1}$ = Mean wage of workers in Firm A

and $\overline{X_2}$ = Mean wage of workers in Firm B

We are given that

$N_1 = 586 \quad N_2 = 648$

$\overline{X_1} = 52.5 \quad \overline{X_2} = 47.5$

∴ Combined Mean, $\overline{X_{12}}$

$$= \frac{(586\times52.5)+(648\times47.5)}{586+648}$$

$$= \frac{61{,}545}{1234} \quad = \text{Rs.}\,49.9$$

5.6.7 Correcting Incorrect Mean

Many a time it happens that we take some wrong items in the data or overlook some items. This results in wrong calculation of Mean. Later we find the correct values and we want to find out correct mean. This can be done using the following steps:

1. Multiply the incorrect mean of the data (incorrect $\overline{X}$) with number of items to find out incorrect $\sum\overline{X}$.
2. Now subtract all the wrong observations from the above values and add the correct observation to the above value to find out correct $\sum\overline{X}$.
3. byNow divide the correct $\sum\overline{X}$. with the number of observations to find correct mean.

Example12. The italics Rs. 98 and Rs. 69 were misread as Rs. 89 and Rs. 96. Find out the correct mean wage.

Solution: We know that

$$\text{Correct} \sum X = \text{Incorrect} \sum X - (\text{Incorrect items}) + (\text{Correct Items})$$

Also $\bar{X} = \frac{\sum X}{N}$

$\Rightarrow$ Incorrect $\sum X = 100 \times 75 = 7500$ ∴ Correct $\sum X = 7500 - (89 + 96) + (98 + 69)$

$= 7482$

$\Rightarrow$ Correct $\bar{X} = \frac{\text{Correct} \sum X}{N}$

$= \frac{7482}{100} \quad = 74.82$

Determination of Missing Frequency

Example 13. Find the missing frequencies of the following series, if $\bar{X} = 33$ and $N = 100$

X	5	15	25	35	45	55
f	5	10	?	30	?	10

Solution: Let the missing frequencies corresponding to $X = 25$ and $X = 45$ be 'f_1' and 'f_2' respectively.

X	f	fX
5	5	25
15	10	150
25	f_1	25f_1
35	30	1050
45	f_2	45f_2
55	10	550
	$\sum f = 55 + f_1 + f_2$	$\sum fX = 1775 + 25f_1 + 45f_2$

Now, $N = 100$ (Given)

∴ $55 + f_1 + f_2 = 100$

$\Rightarrow$ $f_1 + f_2 = 45$ …(i)

Also $\bar{X} = \frac{\sum fX}{N}$

$\Rightarrow$ $33 = \frac{1775 + 25f_1 + 45f_2}{100}$

$\Rightarrow$ $3300 = 1775 + 25f_1 + 45f_2$

$\Rightarrow$ $25f_1 + 45f_2 = 1525$ …(ii)

Solving (i) and (ii), we get

$$25 \times (f_1 + f_2 = 45) \qquad \Rightarrow 25f_1 + 25f_2 = 1125$$

$$1 \times (25f_1 + 45f_2 = 1525) \qquad \Rightarrow 25f_1 + 45f_2 = 1525$$

$$(-) \quad (-) \quad (-)$$

$$-20f_2 = -400$$

$$f_2 = \frac{400}{20} = 20$$

$\therefore \quad f_2 = 20$

Put $\quad f_2 = 20$ in (i)

$f_1 + 20 = 45$

$\Rightarrow \quad f_1 = 45 - 20 = 25$

$\therefore \quad f_1 = 25$

$\therefore \quad f_1 = 25, f_2 = 20$

5.6.8 Properties of Arithmetic mean

1. If we take the deviations of the observations from its Arithmetic mean and then sum up such deviations, then the sum of such deviations will always be zero.
2. If we take the square of the deviations of items from its Arithmetic mean and then sum up such squares, the value obtained will always be less than the square of deviation taken from any other values.
3. If we have a separate mean of two series, we can find the combined mean of the series.
4. If the value of all items in that data is increased or decreased by some constant value say ‘k’, then the Arithmetic mean is also increased or decreased by same ‘k’. In other words, if k is added to the items, then actual mean will be calculated by deducting that k from the mean calculated.
5. If value of all items in the series is divided or multiplied by some constant ‘k’ then the mean is also multiplied or divided by the same constant ‘k’. In other words if we multiply all observations by ‘k’ then actual mean can be calculated by dividing the mean to be obtained by the constant ‘k’.

5.6.10 Limitations of Arithmetic Mean

1. The biggest limitation of the Arithmetic mean is that it is affected by extreme values.
2. If we have an open-end series, it is difficult to measure Arithmetic mean.
3. In the case of qualitative data, it is not possible to calculate the Arithmetic mean.

4. Sometimes it gives an absurd result like we say that there are 20 students in one class and 23 students in other class then the average number of students in a class is 21.5, which is not possible because the students cannot be in fractions.
5. It gives more importance to large-value items than small-value items.
6. Mean cannot be calculated with the help of a graph.
7. It cannot be located by just inspections of the items.

TEST YOUR PROGRESS (A)

1. The following data pertains to the monthly salaries in rupees of the employees of Mohanta Enterprises. Calculate the average salary per employ

3000, 4100, 4700, 5400, 2300, 3400, 3700, 5100, 5300, 4700

2. Calculate the mean for the following data using the shortcut method.

700, 650, 550, 750, 800, 850, 650, 700, 950

3. Following is the height of students in class tenth of a school. Find out the mean height of the students.

Height in Inches	64	65	66	67	68	69	70	71	72	73
No. of students	1	6	10	22	21	17	14	5	3	1

4. Calculate A.M for the following frequency distribution of Marks.

Marks	5	10	15	20	25	30	35	40
No of students	5	7	9	10	8	6	5	2

5. Calculate mean for the following data

Marks	5-15	15-25	25-35	35-45	45-55	55-65
No of Students	8	12	6	14	7	3

6. Calculate mean for the given data by step deviation method

C.I	0-10	10-20	20-30	30-40	40-50	50-60	60-70
f	8	12	14	16	15	9	6

7. From the following data, find the average sale per shop.

Sales in ‘000; units	10-12	13-15	16-18	19-21	22-24	25-27	28-30
No. shops	34	50	85	60	30	15	7

8. For the following data (Cumulative Series) , find the average income .

Income Below in (Rs.)	30	40	50	60	70	80	90
No. of persons	16	36	61	76	87	95	100

9. Calculate the average marks for the following cumulative frequency distribution.

Marks Above	0	10	20	30	40	50	60	70	80	90
No of students	80	77	72	65	55	43	28	16	10	8

10. For a group of 50 male workers, their average monthly wage is Rs. 6300 and for a group of 40 female workers this average is Rs.5400. Find the average monthly wage for the combined group of all the workers.

11. The average mark of 100 students is given to be 45. But later on, it was found that the marks of students getting 64 was misread as 46. Find the correct mean.

12. Find missing frequency when the mean is 35 and number is 68.

X;	0-10	10-20	20-30	30-40	40-50	50-60
F:	4	10	12	?	20	?

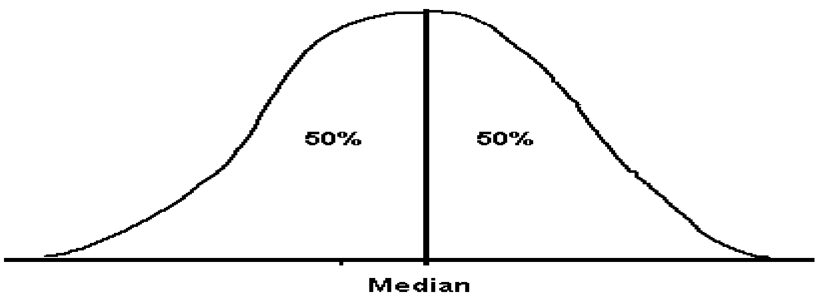
13. The mean age of a combined group of men and women is 30 years. The mean age of a group of men is 32 years and women are 27 years. Find the percentage of men and women in the group

Answers:

1) 4170	**2) 4170**	**3) 68.13 inches**	**4) 20.48**	**5) 31.8**
6) 33.625	**7) 17.8 (in 000 units)**	**8) 48**	**9) 51.75**	**10) 5900**
11) 45.18	**12) 10,12**	**13) Men 60%**		

5.7 MEDIAN

Median is the positional measure of Central tendency. It means the median does not depend upon the value of the item under the observation, rather it depends on the position of the item in the series. Median is a value that divides the series exactly into two equal parts, it means 50% of the observations lie below the median and 50% of the observations lie above the median. However, it is important to arrange the series either in ascending order or in descending order before calculation of the Median. If the series is not arranged, then Median cannot be calculated



For calculating Median

1. Series should be in ascending or descending order
2. Series should be exclusive, not inclusive

5.7.1 Median in case of Individual series.

For calculating the median in individual series, following are the steps:

1. Arrange the series in ascending or descending order.
2. Calculate the number of observations. It is denoted by N.
3. Calculate the $\left(\frac{N+1}{2}\right)^{th}$ term
4. Corresponding value to this item is the median of the data
5. In case there are even number of items in the series, this value will be in fraction. In that case take the arithmetic mean of the adjacent items in which Median is falling. For example, if it is 4.5 than take arithmetic mean of 4^{th} item and 5^{th} item.

$$\textbf{Median = value of } \left(\frac{N+1}{2}\right)^{th} \textbf{ term}$$

When the number of observations N is odd

Example 20. Calculation median from the following observations:

$$\mathbf{15, 17, 19, 22, 18, 47, 25, 35, 21}$$

Solution: Arranging the given items in ascending order, we get

$15, 17, 18, 19, 21, 22, 25, 35, 47$

Now Median, $M = \text{Size of } \left(\frac{N+1}{2}\right)^{th}$ item

$M = \text{Size of } \left(\frac{9+1}{2}\right)^{th}$ item

$= \text{Size of } 5^{th}$ item

$= 21$

$\Rightarrow \quad M = 21$

When the number of observations N is even

Example 21. Find the median from the following data

$\mathbf{28, 26, 24, 21, 23, 20, 19, 30}$

Solution: Arranging the given figures in ascending order, we get

$19, 20, 21, 23, 24, 26, 28, 30$

Now Median, $M = \text{Size of } \left(\frac{N+1}{2}\right)^{th}$ item

$M = \text{Size of } \left(\frac{8+1}{2}\right)^{th}$ item

$= \text{Size of } 4.5^{th}$ item

$= \frac{4^{th} \text{ item} + 5^{th} \text{ item}}{2}$

$= \frac{23+24}{2} = \frac{47}{2} = 23.5$

$\Rightarrow \quad M = 23.5$

5.7.2 Median in case of Discrete series

Following are the steps in case of discrete series:

1. Arrange the data in ascending or descending order.
2. Find the cumulative frequency of the series.
3. Find the $\left(\frac{N+1}{2}\right)^{th}$ term
4. Now look at this term in the cumulative frequency of the series.
5. Value against which such cumulative frequency falls is the median value.

$$\textbf{Median = value of } \left(\frac{N+1}{2}\right)^{th} \textbf{ term}$$

Example 22. Calculate the value of median, if the data is as given below:

Height (in cms.)	110	125	250	200	150	180

No. of Students	8	12	3	10	13	15

Solution: Arranging the given data in ascending order, we get

Height (in cms.)	No. of Students f	Cumulative Frequency C · f
110	8	8
125	12	20
150	13	33
180	15	48
200	10	58
250	3	61
	$\sum f = N = 61$	

Now Median, $M = \text{Size of } \left(\frac{N+1}{2}\right)^{th}$ item

$$M = \text{Size of } \left(\frac{6+1}{2}\right)^{th} \text{ item}$$

$$= \text{Size of } 31^{st} \text{ item}$$

$$= 150$$

⇒ Median, $M = 150$ cms.

5.7.3 Median in case of Continuous Series

Following are the steps in case of continuous series:

1. Arrange the data in ascending or descending order.
2. Find the cumulative frequency of the series.
3. Find the $\left(\frac{N}{2}\right)^{th}$ term
4. Now look at this term in the cumulative frequency of the series. The value equal to or higher than term calculated in third step is the median class.
5. Find median using following formula.
6. $\mathbf{M = L + \frac{\frac{N}{2} - C \cdot f}{f} \times i}$
7. Where M = Median
8. L = Lower Limit of Median Class
9. N = Number of Observations.
10. c.f. = Cumulative frequency of the Median Class.
11. f = Frequency of the class preceding Median Class.

12. i = Class interval of Median Class

Example 23. Calculate Median

Marks	5-10	10-15	15-20	20-25	25-30	30-35
No. of Students	8	7	14	16	9	6

Solution:

C. I.	No. of Students f	Cumulative Frequency C · f
5-10	8	8
10-15	7	15
15-20	14	29
20-25	16	45
25-30	9	54
30-35	6	60
	$\sum f = N = 60$	

Median, $M = \text{Size of } \left(\frac{N}{2}\right)^{th}$ item

$M = \text{Size of } \left(\frac{60}{2}\right)^{th}$ item

$= \text{Size of } 30^{th}$ item

$\Rightarrow$ Median lies in the class interval $20 - 25$

As Median, $M = L + \frac{\frac{N}{2} - C \cdot f}{f} \times i$

Here L = Lower limit of the median class $= 20$

$N = 60$

$C \cdot f = 29$

$f = 16$

i = Class – length of the median class $= 5$

$\therefore$ $M = 20 + \frac{(30-29)}{16} \times 5$

$= 20 + \frac{5}{16}$

$= 20 + 9.312 = 29.312$

$\Rightarrow$ $M = 29.312$

Inclusive Series – It must be converted to Exclusive Series before calculation of the Median.

Example 24. Find Median from the given data

X	10-19	20-29	30-39	40-49	50-59	60-69	70-79	80-89

f	6	53	85	56	21	16	4	4

Solution: Converting the given data into exclusive form, we get

$$\left[\text{Correction factor} = \frac{L_2 - U_1}{2} = \frac{20 - 19}{2} = \frac{1}{2} = 0.5\right]$$

(0.5 is subtracted from all lower limits and added to all upper limits)

X	f	Cumulative frequency $C \cdot f$
9.5-19.5	6	6
19.5-29.5	53	59
29.5-39.5	85	144
39.5-49.5	56	200
49.5-59.5	21	221
59.5-69.5	16	237
69.5-79.5	4	241
79.5-89.5	4	245
	$\sum f = N = 245$	

Median, $M = \text{Size of } \left(\frac{N}{2}\right)^{th}$ item

$M = \text{Size of } \left(\frac{245}{2}\right)^{th}$ item

$= \text{Size of } 122.5^{th}$ item

∴ The real class limits of the median class $= (29.5 - 39.5)$

So $M = L + \frac{\left(\frac{N}{2} - C \cdot f\right)}{f} \times i$

⇒ $M = 29.5 + \left(\frac{122.5 - 59}{85}\right) \times 10$

$= 29.5 + \left(\frac{63.5}{85} \times 10\right)$

$= 29.5 + \left(\frac{635}{85}\right)$

$= 29.5 + 7.47 = 36.97$

⇒ $M = 36.97$

Cumulative Series (More than and less than)

Example 25. Find median, if the data is as given below:

Marks More than	20	35	50	65	80	95
No. of Students	100	94	74	30	4	1

Solution: Converting the given data into class – interval form, we get

Marks (C. I.)	Frequency (f)	Cumulative Frequency (C · f)
20-35	100-94=6	6
35-50	94-74=20	26
50-65	74-30=44	70
65-80	30-4=26	96
80-95	4-1=3	99
95-110	1	100
	$\sum f = N = 100$	

Now Median, $M = \text{Size of } \left(\frac{N}{2}\right)^{th} \text{ item}$

$M = \text{Size of } \left(\frac{100}{2}\right)^{th} \text{ item}$

$= \text{Size of } 50^{th} \text{ item}$

$\Rightarrow$ Median lies in the class interval $= 50 - 65$

So $M = L + \frac{\left(\frac{N}{2} - C \cdot f\right)}{f} \times i$

$\Rightarrow$ $M = 50 + \left(\frac{50-26}{44}\right) \times 15$

$= 50 + \left(\frac{24}{44} \times 15\right)$

$= 50 + 8.18 = 58.18$

$\Rightarrow$ $M = 58.18$

Example 26. Find median, if the data is as given below:

Marks Less than	10	20	30	40	50	60	70	80
No. of Students	20	30	50	94	96	127	198	250

Solution: Converting the given data into class interval form, we get

Marks (C.I.)	No. of Students (f)	Cumulative Frequency (C·f)
0-10	20	20
10-20	30-20=10	30
20-30	50-30=20	50
30-40	94-50=44	94
40-50	96-94=2	96
50-60	127-96=31	127

60-70	198-127=71	198
70-80	250-198=52	250
	$\sum f = N = 250$	

Now Median, $M = \text{Size of} \left(\frac{N}{2}\right)^{th}$ item

$$M = \text{Size of} \left(\frac{250}{2}\right)^{th} \text{ item}$$

$$= \text{Size of } 125^{th} \text{ item}$$

⇒ Median lies are the class – interval $= 50 - 60$

So $M = L + \frac{\frac{N}{2} - C \cdot f}{f} \times i$

⇒ $M = 50 + \left(\frac{125-96}{31}\right) \times 10$

$$= 50 + \left(\frac{29}{31} \times 10\right)$$

$$= 50 + \frac{290}{31}$$

$$= 50 + 9.35 = 59.35$$

⇒ $M = 59.35$

Mid – Value Series

Example 27. Find the value of median for the following data:

Mid Value	15	25	35	45	55	65	75	85	95
f	8	26	45	72	116	60	38	22	13

Solution: It is clear from the mid – value that the class size is 10. For finding the limits of different classes, apply the formula:

$$L = m - \frac{i}{2} \quad \text{and} \quad U = m + \frac{i}{2}$$

Where, L and U denote the lower and upper limits of different classes, 'm' denotes the mid – value of the corresponding class interval and 'i' denotes the difference between mid values.

∴ Corresponding to mid – value '15', we have

$$L = 15 - \frac{10}{2} \quad \text{and} \quad U = 15 + \frac{10}{2}$$

i.e. $C.I. = 10 - 20$

Similarly other class intervals can be located

Mid Value	f	C. I.	Cumulative Frequency $C \cdot f$
15	8	10-20	8

25	26	20-30	34
35	45	30-40	79
45	72	40-50	151
55	116	50-60	267
65	60	60-70	327
75	38	70-80	365
85	22	80-90	387
95	13	90-100	400
	$N = 100$		

Now Median, $M = \text{Size of } \left(\frac{N}{2}\right)^{\text{th}}$ item

$$M = \text{Size of } \left(\frac{400}{2}\right)^{\text{th}} \text{ item}$$

$$= \text{Size of } 200^{\text{th}} \text{ item}$$

$\Rightarrow$ Median lies is the class – interval $= 50 - 60$

So $M = L + \frac{\frac{N}{2} - C{\cdot}f}{f} \times i$

$\Rightarrow$ $M = 50 + \left(\frac{200-151}{116}\right) \times 10$

$$= 50 + \left(\frac{49}{116} \times 10\right)$$

$$= 50 + \frac{490}{116}$$

$$= 50 + 4.224 = 54.224$$

$\Rightarrow$ $M = 54.224$

Determination of Missing Frequency

Example 28. The median value for the following frequency distribution is 35 and the sum of all the frequency is 170. Using the formula for median, find the missing frequencies.

C.I.	0-10	10-20	20-30	30-40	40-50	50-60	60-70
f	**10**	**20**	**a**	**40**	**b**	**25**	**15**

Solution: Let the missing frequencies be f_1, f_2 and f_3 respectively.

C. I.	f	Cumulative Frequency (C · f)
$0 - 10$	10	10
$10 - 20$	20	10+20= 30
$20 - 30$	a	30 + a
$30 - 40$	40	30 +a +40 = 70 + a
$40 - 50$	b	70 +a + b + 25 = 95+a+b
$50 - 60$	25	110+a+b

$60-70$	15	$46+f_1+f_2$
	$\sum f = 110+a+b$	

Given that sum of frequencies is N = 170.

∴ 110 + a + b = 170

⇒ a + b = 170 – 110 = 60.

⇒ a + b = 60 … (1)

Now, N /2 = 170/ 2 = 85.

$N/$ 2 = 170/ 2 = 85.

Given that median is 35 which lines in the class 30 – 40.

Hence, median class is 30 – 40.

∴ l = Lower limit of median class = 30.

h = Class interval = 40 – 30 = 10.

f = Frequency of the median class = 40.

cf = Cumulative frequency of the class before median class = 30 + a.

$M = L + \frac{\frac{N}{2} - C{\cdot}f}{f} \times i$

⇒ 35 = $30 + \frac{85-30-a}{40} \times 10$

⇒ 35 = $30 + \frac{55-a}{4}$

⇒ 35 = $\frac{120+\ 55-a}{4}$

⇒ 175 – a = 140

⇒ a = 175 – 140 = 35.

Now, putting a = 35 in equation (1),

we get 35 + b = 60

⇒ b = 60 – 35 = 25.

Hence, the missing frequencies are a = 35 and b = 25.

5.7.5 Limitations of Median

1. It is not capable of further algebraic treatment.
2. It is positional average and is not based on all observation.

3. It is very much affected by fluctuation in sampling.
4. Median needs arrangement of data before calculation.
5. In case of continuous series, it assumes that values are equally distributed in a particular class.

TEST YOUR PROGRESS (C)

1. Calculate Median

30, 45,75, 65, 50, 52, 28, 40, 49, 35, 52,

2. Calculate Median

36, 32,28, 22, 26, 20, 18, 40,

3. Find Median

Wages:	100	150	80	200	250	180
No. of workers	24	26	16	20	6	30

4. Calculate Median

X;	0-5	5-10	10-15	15-20	20-25	25-30	30-35
F:	4	6	10	16	12	8	4

5. Calculate Median:

X;	10-19	20-29	30-39	40-49	50-59	60-69
F:	4	8	12	16	10	6

6. Find Median:

Income	100-200	200-400	400-700	700-1200	1200-2000
Number of firms	40	100	260	80	20

7. Find missing frequency when median is 50 and number is 100.

X;	0-20	20-40	40-60	60-80	80-100
F:	1 4	?	27	?	15

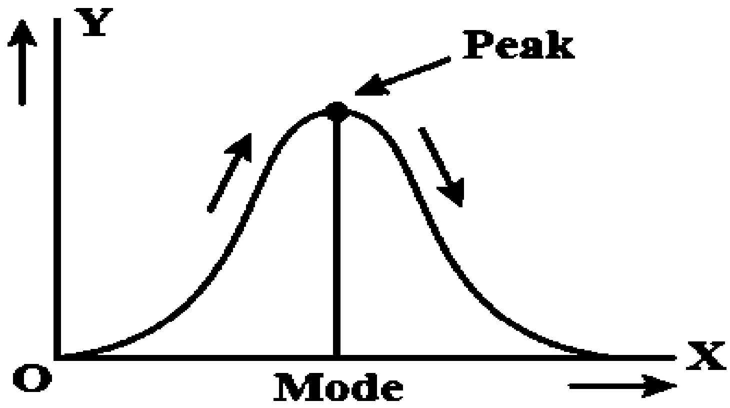
Answers:

1) 49	**2) 27**	**3) 150**
4) 18.125	**5) 42**	**6) 526.92**
7) 23,21		

5.8 MODE

Mode is another positional measure of Central Tendency. Mode is the value that is repeated most number of time in the series. In other words, the value having highest frequency is called Mode.

The term 'Mode' is taken from French word 'La Mode' which means the most fashionable item. So, Mode is the most popular item of the series.



For calculating Mode

1. Series should be in ascending or descending order.
2. Series should be exclusive, not inclusive.
3. Series should have equal class intervals.

5.8.1 Mode in Individual Series

In case of Individual series, following are the steps of finding the Mode.

1. Arrange the series either in ascending order or descending order.
2. Find the most repeated item.
3. This item is Mode.

Example 32. Calculate mode from the following data of marks obtained by 10 students

S. No.	1	2	3	4	5	6	7	8	9	10
Marks obtained	10	27	24	12	27	27	20	18	15	30

Solution: By Inspection

It can be observed that 27 occur most frequently i.e. 3 times. Hence, mode = 27 marks

By converting into discrete series

Marks Obtained	Frequency
10	1
12	1
15	1
18	1
20	1
24	1
27	3

30	1
	$N = 10$

Since, the frequency of 27 is maximum i. e. 3

It implies the item 27 occurs the maximum number of times. Hence the modal marks are 27.

$$\text{Mode} = 27$$

5.8.2 Mode in discrete series

In case of discrete series, we can find mode by two methods that are Observation Method and Grouping Method.

Observation Method: Under this method value with highest frequency is taken as mode.

Grouping Method: Following are the steps of Grouping method:

- Prepare a table and put all the values in the table in ascending order.
- Put all the frequencies in first column. Mark the highest frequency.
- In second column put the total of frequencies taking two frequencies at a time like first two, then next two and so on. Mark the highest total.
- In third column put the total of frequencies taking two frequencies at a time but leaving the first frequency like second and third, third and fourth and so on. Mark the highest total.
- In fourth column put the total of frequencies taking three frequencies at a time like first three, than next three and so on. Mark the highest total.
- In fifth column put the total of frequencies taking three frequencies at a time but leaving the first frequency like second, third and fourth; than fifth, sixth and seventh and so on. Mark the highest total.
- In sixth column put the total of frequencies again taking three frequencies at a time but leaving the first two frequencies. Mark the highest total.
- Value that is marked highest number of times is the mode.

Example 33. Find the modal value for the following distribution

Age (in years)	8	9	10	11	12	13	14	15
No. of Persons	5	6	8	7	9	8	9	6

Solution: Here, as maximum frequency 9 belongs to two age values 12 and 14, so its not possible to determine mode by inspection. We will have to determine the modal value through grouping and analysis table.

<table>
<tr><th colspan="7">Grouping Table</th></tr>
<tr><th rowspan="2">Age (in years)</th><th colspan="6">Frequency</th></tr>
<tr><th>G_1</th><th>G_2</th><th>G_3</th><th>G_4</th><th>G_5</th><th>G_6</th></tr>
<tr><td>8</td><td>5</td><td rowspan="2">11</td><td rowspan="3">14</td><td rowspan="3">19</td><td rowspan="4">21</td><td></td></tr>
<tr><td>9</td><td>6</td><td rowspan="3">24</td></tr>
<tr><td>10</td><td>8</td><td rowspan="2">15</td></tr>
<tr><td>11</td><td>7</td><td rowspan="2">16</td><td rowspan="4">24</td></tr>
<tr><td>12</td><td>9</td><td rowspan="2">17</td><td rowspan="3">26</td><td rowspan="4">23</td></tr>
<tr><td>13</td><td>8</td><td rowspan="3">17</td></tr>
<tr><td>14</td><td>9</td><td rowspan="2">15</td></tr>
<tr><td>15</td><td>6</td><td></td><td></td></tr>
</table>

Analysis Table								
Group No.	8	9	10	11	12	13	14	15
G_1					×		×	
G_2					×	×		
G_3						×	×	
G_4				×	×	×		
G_5					×	×	×	
G_6			×	×	×			
Total	×	×	1	2	5	4	3	×

Since, 12 occurs maximum number of times i. e. 5 times, the modal age is 12 years

$$\text{Mode} = 12$$

5.8.3 Mode in Continuous series

In the case of continuous series, we can find mode by two methods that are Observation Method and Grouping Method.

1. **Observation Method**: Under this method value with the highest frequency is taken as mode class then the mode formula is applied which is given below.
2. **Grouping Method**: The following are the steps of Grouping method:
   - Prepare a table and put all the classes of data in the table in ascending order.
   - Put all the frequencies in first column. Mark the highest frequency.
   - In the second column put the total of frequencies taking two frequencies at a time like the first two, then next two, and so on. Mark the highest total.

- In the third column put the total of frequencies taking two frequencies at a time but leaving the first frequency like second and third, third and fourth and so on. Mark the highest total.
- In the fourth column put the total of frequencies taking three frequencies at a time like the first three, then next three and so on. Mark the highest total.
- In the fifth column put the total of frequencies taking three frequencies at a time but leaving the first frequency like second, third and fourth; then fifth, sixth and seventh and so on. Mark the highest total.
- In sixth column put the total of frequencies again taking three frequencies at a time but leaving the first two frequencies. Mark the highest total.
- The class that is marked highest number of times is the mode class.
- Apply the following formula for calculating the mode:

$$\mathbf{Z = L + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times i}$$

Where, Z = Mode, L = Lower limit of the mode class, f_m = Frequency of mode class.

f_1 = Frequency of class proceeding mode class, f_2 = Frequency of class succeeding mode class

i = Class interval

Example 34. Find the mode for the following frequency distribution

Age (in years)	30-35	35-40	40-45	45-50	50-55	55-60
No. of Persons	3	8	12	20	15	2

Solution: Here, the maximum frequency is corresponding to the class – interval $45 - 50$.

So, the modal class is $45 - 50$.

Now, the mode is given by the formula

Mode, $Z = L + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times i$

Here $L =$ Lower limit of modal class $= 45$

$f_m =$ Frequency of modal class $= 20$

$f_1 =$ Frequency of class preceeding the modal class $= 12$

$f_2 =$ Frequency of class succeeding the modal class $= 15$

$i =$ Class length of modal class $= 5$

$\therefore$ Mode, $Z = 45 + \frac{20-12}{(2\times20)-12-15} \times 5$

$$= 45 + \frac{8}{40-27} \times 5$$

$$= 45 + 3.07$$

$$= 48.1 \text{ years (approx.)}$$

$\Rightarrow \quad Z = 48.1$ year

Example 35. Calculate mode from the following data

C.I.	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90	90-100
f	2	9	10	13	11	6	13	7	4	1

Solution: Here as it is not possible to find modal class by inspection, so we have to determine it through grouping and analysis table.

<table>
<tr><th colspan="7">Grouping Table</th></tr>
<tr><th rowspan="2">C. I.</th><th colspan="6">Frequency</th></tr>
<tr><th>G_1</th><th>G_2</th><th>G_3</th><th>G_4</th><th>G_5</th><th>G_6</th></tr>
<tr><td>$0-10$</td><td>2</td><td rowspan="2">11</td><td></td><td rowspan="3">21</td><td></td><td></td></tr>
<tr><td>$10-20$</td><td>9</td><td rowspan="2">19</td><td rowspan="3">32</td><td></td></tr>
<tr><td>$20-30$</td><td>10</td><td rowspan="2">23</td><td rowspan="3">34</td></tr>
<tr><td>$30-40$</td><td>13</td><td rowspan="2">24</td><td rowspan="3">30</td></tr>
<tr><td>$40-50$</td><td>11</td><td rowspan="2">17</td><td rowspan="3">30</td></tr>
<tr><td>$50-60$</td><td>6</td><td rowspan="2">19</td><td rowspan="3">26</td></tr>
<tr><td>$60-70$</td><td>13</td><td rowspan="2">20</td><td rowspan="3">24</td></tr>
<tr><td>$70-80$</td><td>7</td><td rowspan="2">11</td><td rowspan="3">12</td></tr>
<tr><td>$80-90$</td><td>4</td><td rowspan="2">5</td><td></td></tr>
<tr><td>$90-100$</td><td>1</td><td></td><td></td><td></td></tr>
</table>

Analysis Table

Group No.	**0-10**	**10-20**	**20-30**	**30-40**	**40-50**	**50-60**	**60-70**	**70-80**	**80-90**	**90-100**
G_1				×			×			
G_2			×	×						
G_3				×	×					
G_4				×	×	×				
G_5		×	×	×						
G_6			×	×	×					
Total	×	1	3	6	3	1	1	×	×	×

Clearly the modal class is $30-40$

Now the mode is given by the formula

$$\text{Mode, } Z = L + \frac{f_m - f_1}{2f_m - f_1 - f_2} \times i$$

Here L = Lower limit of modal class $30-40 = 30$

f_m = Frequency corresponding to modal class $= 13$

f_1 = Frequency of interval preceding modal class

f_2 = Frequency of interval succeeding and

i = Class length of modal class

∴ Mode, $Z = 30 + \frac{13-10}{(2\times13)-10-11} \times 10$

$= 30 + \frac{3}{26-21} \times 10$

$= 30 + \frac{30}{5}$

$= 30 + 6$

$= 36$

⇒ $Z = 36$

Example 36. Determine the missing frequencies when it is given that $N = 230$, Median, $M = 233.5$ and Mode, $Z = 234$

C.I	200-210	210-220	220-230	230-240	240-250	250-260	260-270
f	4	16	-	-	-	6	4

Solution: Let the missing frequencies be f_1, f_2 and f_3 respectively.

C.I	f	C · f
$200-210$	4	4
$210-220$	16	20
$220-230$	f_1	$20+f_1$
$230-240$	f_2	$20+f_1+f_2$
$240-250$	f_3	$20+f_1+f_2+f_3$
$250-260$	6	$26+f_1+f_2+f_3$
$260-270$	4	$30+f_1+f_2+f_3$
	$N = 230 = \sum f$ $\sum f = 30+f_1+f_2+f_3$	

Now $N = 230 = \sum f$ (Given)

$= 30 + f_1 + f_2 + f_3$

⇒ $f_1 + f_2 + f_3 = 230 - 30 = 200$

⇒ $f_1 + f_2 + f_3 = 200$ …(i)

Also, Median $= 233.5$ (Given)

⇒ Median class is $230 - 240$

⇒ $M = L + \frac{\frac{N}{2} - C\cdot f}{f} \times i$

$$233.5 = 230 + \frac{\frac{230}{2} - (20 + f_1)}{f_2} \times 10$$

$$3.5 = \frac{115 - 20 - f_1}{f_2} \times 10$$

$$3.5f_2 = 950 - 10f_1$$

$\Rightarrow$ $10f_1 + 3.5f_2 = 950$ …(ii)

Now Mode $= 234$ lies in $230 - 240$

$\therefore$ $Z = L + \frac{f_2 - f_1}{2f_2 - f_1 - f_3} \times i$

$\Rightarrow$ $234 = 230 + \frac{f_2 - f_1}{2f_2 - f_1 - f_3} \times 10$

$\Rightarrow$ $4 = \frac{f_2 - f_1}{2f_2 - f_1 - (200 - f_1 - f_2)} \times 10$ [Using (i)]

$\Rightarrow$ $4 = \frac{f_2 - f_1}{2f_2 - f_1 - 200 - f_1 - f_2} \times 10$

$\Rightarrow$ $4 = \frac{(f_2 - f_1) \times 10}{3f_2 - 200}$

$\Rightarrow$ $12f_2 - 800 = 10f_2 - 10f_1$

$\Rightarrow$ $2f_2 - 800 + 10f_1 = 0$

$\Rightarrow$ $10f_1 + 2f_2 = 800$ …(iii)

Solving (ii) and (iii), we get

$$10f_1 + 3.5f_2 = 950$$

$$10f_1 + 2f_2 = 800$$

$$(-) \quad (-) \quad (-)$$

$$1.5f_2 = 150$$

$\Rightarrow$ $f_2 = \frac{150}{1.5} = 100$

$f_2 = 100$ …(iv)

Put (iv) in (iii)

$$10f_1 + 2(100) = 800$$

$\Rightarrow$ $10f_1 = 800 - 200 = 600$

$\Rightarrow$ $10f_1 = 600$

$\Rightarrow$ $f_1 = 60$ …(v)

Put (iv) and (v) in (i)

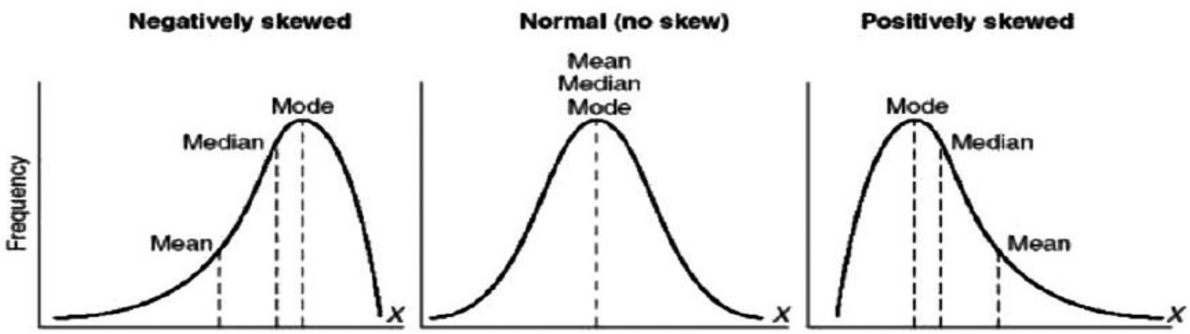
$$60 + 100 + f_3 = 200$$

⇒ $f_3 = 40$

∴ The missing frequencies are 60, 100 and 40.

5.9 RELATION BETWEEN MEAN, MEDIAN AND MODE

In a normal series the value of Mean, Median and Mode is always same. However, Karl Pearson studied the empirical relation between the Mean, Median and Mode and found that in moderately skewed series the Median always lies between the Mean and the Mode. Normally it is two third distance from Mode and one third distance from Mean.



On the basis of this relation following formula emerged

Mode = 3 Median – 2 Mean
or $\boldsymbol{Z = 3M - 2\bar{X}}$

Example. Calculate M when $\bar{X}$ and Z of a distribution are given to be 35.4 and 32.1 respectively.

Solution: We are given that

Mean, $\bar{X} = 35.4$

Mode, $Z = 32.1$

As we know the empirical relation between Mean, Median and Mode.

i.e. Mode = 3 Median – 2 Mode

⇒ $Z = 3M - 2\bar{X}$

⇒ $M = \frac{1}{3}\left(Z + 2\bar{X}\right)$

⇒ $M = \frac{1}{3}(32.1 + 2(35.4))$

$= \frac{1}{3}(32.1 + 70.8)$

$= \frac{1}{3}(102.9) = 34.3$

⇒ Median, $M = 34.3$

TEST YOUR PROGRESS - D

1. Find Mode:

X 22, 24, 17, 18, 19, 18, 21, 20, 21, 20, 23, 22, 22, 22

2. Find Mode by inspection method

X	6	12	18	24	30	36	42	48
f	9	11	25	16	9	10	6	3

3. Find Mode by Grouping Method

X	21	22	25	26	27	28	29	30
F	7	10	15	18	13	7	3	2

4. Find Mode by Grouping Method and inspection method

X;	0-10	10-20	20-30	30-40	40-50	50-60	60-70	70-80
F:	2	18	30	45	35	20	6	4

5. Calculate mode using grouping and analysis methods.

X	100-110	110-120	120-130	130-140	140-150	150-160	160-170	170-180
f	4	6	20	32	33	17	8	2

6. Find Mode

X	0-100	100-200	200-400	400-500	500-700
F:	5	15	40	32	28

Answers:

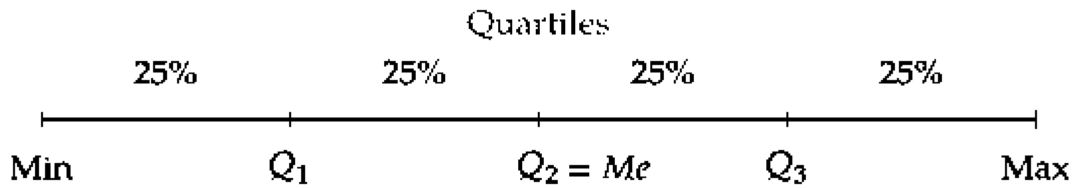
1) 22	**2) 18**	**3) 26**
4) 36	**5) 56.46**	**6) 440**

5.10 OTHER POSITIONAL MEASURES (QUARTILES, DECILES AND PERCENTILES)

As median divide the series into two equal parts, there are many other positional measures also. These Positional measures are also known as partition values. Following are some of the positional measure

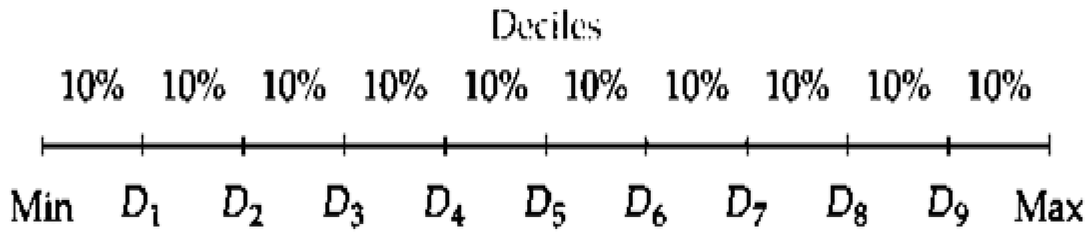
A) Quartiles

Quartile are the values that divide the series in four equal parts. There are total three quarter in number denoted by Q_1, Q_2 and Q_3. First quartile is placed at 25% of the items, second quartile at 50% of the items, third quartile at 75% of the items. The value of second quartile is always equal to Median.



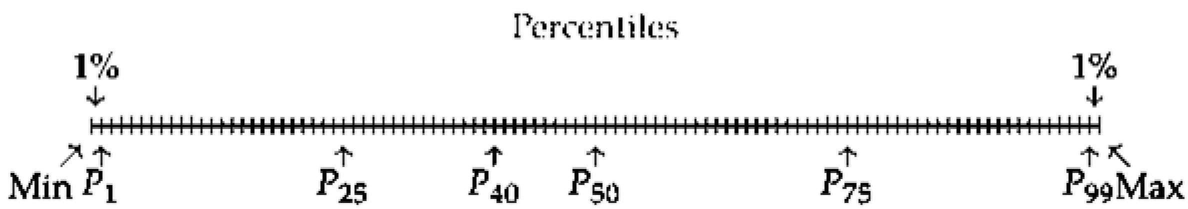
B) Deciles

Deciles are the values that divide the series in ten equal parts. There is total nine Deciles in number denoted by D_1, D_2, D_3 and so on upto D_9. The first decile is placed at 10% of the items, second quartile at 20% of the items, similarly last at 90% of the items. The value of fifth Decile is always equal to Median.



C) Percentile

Percentiles are the values that divide the series in hundred equal parts. There is total ninety-nine Percentiles in number denoted by P_1, P_2, P_3 and so on upto P_{99}. The first Percentile is placed at 1% of the items, second quartile at 2% of the items, similarly at 99% of the items. The value of fiftieth Percentile is always equal to Median.



The methods of finding positional measures are same as in case of median. However, following are the formulas that can be used for finding positional measures.

5.11 SUM UP

- Average is the value that represent its series.
- A good average has many characteristics.
- Average is also known as Central Tendency.
- There are mainly five types of average Arithmetic Mean, Geometric Mean, Harmonic Mean, Median, Mode.

- Arithmetic mean is most popular average.
- Median divide the series in two equal parts.
- Mode is value repeated most number of time.
- There are other positional measures like Quartile, Decile and Percentile.

5.12 KEY TERMS

- **Average:** Average is the single value which is capable of representing its series. It is the value around which other values in the series move.
- **Arithmetic Mean:** "The value obtained by dividing sum of observations with the number of observations".
- **Median:** It is a value that divide the series in two equal parts.
- **Mode:** It is the most repeated value of the series.
- **Quartile:** It is a value that divide the series in four equal parts.
- **Decile:** It is a value that divide the series in ten equal parts.
- **Percentile:** It is a value that divide the series in hundred equal parts.

5.13 QUESTIONS FOR PRACTICE

Short Answer Type

Q1. What is central tendency?

Q2. What is arithmetic mean?

Q3. How you can calculate combined arithmetic mean.

Q4. What is the median?

Q5. What is mode?

Q6. What is Quartile, Percentile and Deciles?

Q7. According to you which measure of average is best.

Long Answer Type

Q1. What are the uses of measuring central tendency?

Q2. Give features of ideal measure of average.

Q3. Give properties, advantages and limitations of Arithmetic mean.

Q4. Give merits and limitations of Median.

Q5. How Mode is calculated for grouping method. Give its merits and limitations.

Q6. Explain relation between Mean, Median and Mode.

Q7. What is positional average? Give various positional average.

5.14 FURTHER READINGS

- J. K. Sharma, Business Statistics, Pearson Education.
- S.C. Gupta, Fundamentals of Statistics, Himalaya Publishing House.
- S.P. Gupta and Archana Gupta, Elementary Statistics, Sultan Chand and Sons, New Delhi.
- Richard Levin and David S. Rubin, Statistics for Management, Prentice Hall of India, New Delhi.

BACHELOR OF ARTS

SEMESTER – VI

QUANTITATIVE M ETHODS

UNIT 6: MEASURES OF DISPERSION AND SKEWNESS

STRUCTURE

6.0 OBJECTIVES

After studying the Unit, students will be able to:

- Explain the meaning of Dispersion
- Compare absolute and relative measures of Dispersion
- Understand features of a good measure of Dispersion
- Calculate the Range and Quartile Deviation
- Measure the Dispersion using Mean and Standard Deviation
- Compare the variation of the two series

6.1 INTRODUCTION AND MEANING

Statistics is a tool that helps us in the extraction of information from a large pool of data. Many tools in statistics help us in the extraction of data. Central tendency of data is one such tool. A good measure of central tendency could represent the whole data. However, many a time we find that the average is not representing it data. The following example will make this clear:

Series X	Series Y	Series Z
100	94	1
100	105	2
100	101	3

100 100	98 102	4 490
$\sum X = 500$	$\sum Y = 500$	$\sum Z = 500$
$\bar{X} = \frac{\sum X}{N} = \frac{500}{5} = 100$	$\bar{Y} = \frac{\sum Y}{N} = \frac{500}{5} = 100$	$\bar{Z} = \frac{\sum Z}{N} = \frac{500}{5} = 100$

We can see that in all the above series the average is 100. However, in the first series average fully represents its data as all the items in the series are 100 and the average is also 100. In the second series, the items are very near to its average which is 100, so we can say that average is a good representation of the series. But in case of third series, average is not represent its data as there is a lot of difference between items and the average. To understand the nature of data it is very important to see the difference between items and the data. This could be done by using dispersion. Dispersion is a very important statistical tool that helps us in progress the nature of data. Dispersion shows the extent to which individual items in the data differ from its average. It is a measure of the difference between data and the individual items. It indicates how that lacks uniformity. Following are some of the definitions of Dispersion.

According to Spiegel, "The degree to which numerical data lend to spread about an average value is called the variation, or dispersion of the data". As the dispersion gives average of difference between items and their Central tendency, it is also known as average of the second order.

6.2 BENEFITS / USES OF DISPERSION

The benefits of Dispersion analysis are outlined as under:

1. **To examine the reliability of Central tendency:** We have already discussed that a good measure of Central tendency could represent its series. Dispersion gives us the idea of whether average is in a position to represent its series or not. Based on this we can calculate the reliability of the average.
2. **To compare two series**: In case there are two series and we want to know which series has more variation, we can use dispersion as its tool. In such cases normally we use a relative measure of dispersion for comparing two series.
3. **Helpful in quality control**: Dispersion is a tool that is frequently used in quality control by the business houses. Every manufacturer wants to maintain same quality and reduce the variation in production. Dispersion can help us in finding the deviations and removing the deviations in quality.

4. **Base of further statistical analysis:** Dispersion is a tool that is used in some statistical analyses. For example, we use dispersion while calculating correlation, Regression, Skewness and Testing the Hypothesis, etc.

6.3 FEATURES OF GOOD MEASURE OF DISPERSION

A good measure of dispersion has some features which are mentioned below:

- A good tool of dispersion must be easy to understand and simple to calculate.
- A good measure of dispersion must be based on all the values in the data.
- It should not be affected by the presence of extreme values in the data.
- A good measure is rigidly defined.
- A good measure of dispersion must be capable of further statistical analysis.
- A good measure must not be affected by the sampling size.

6.4 ABSOLUTE AND RELATIVE MEASURE OF DISPERSION

Two measures of dispersion are absolute measure and relative measure:

1. **Absolute measure:** the absolute measure of dispersion is expressed in the same statistical unit in which the original values of that data are expressed. For example, if the original data is represented in kilograms, the dispersion will also be represented in kilograms. Similarly, if data is represented in rupees the dispersion will also be represented in rupees. However, this measure is not useful when we have to compare two or more series that have different units of measurement or belong to different populations.
2. **Relative measure of Dispersion**: The relative measure of dispersion is independent of the unit of measurement and is expressed in pure numbers. Normally it is a ratio of the dispersion to the average of the data. It is very useful when we have to compare two different series that have different units of measurement or belong to different populations.

Absolute Measure of Dispersion

- Range
- Quartile Deviation
- Mean Deviation
- Standard Deviation

Relative Measure of Dispersion

- Coefficient of Range

- Coefficient of Quartile Deviation
- Coefficient of Mean Deviation
- Coefficient of Standard Deviation

6.5 MEASURE OF DISPERSION - RANGE

The range is one of the simplest and oldest measures of Dispersion. We can define Range as the difference between highest value of the data and the lowest value of the data. The more is the difference between highest and the lowest value, more is the value of Range which shows high dispersion. Similarly, less is the difference between the highest and lowest value, and less is value of the Range that shows less dispersion. Following is a formula for calculating the value of range:

Range = Highest Value - Lowest Value

R = H – L

Coefficient of Range: Coefficient of Range is a relative measure of Range and can be calculated using the following formula.

Coefficient of Range = $\frac{\textbf{Highest Value} - \textbf{Lowest Value}}{\textbf{Highest Value} + \textbf{Lowest Value}} = \frac{\mathbf{H - L}}{\mathbf{H + L}}$

6.5.1 Range in Individual Series:

Example 1. Following are the daily wages of workers, find out the value of Range and Coefficient of Range.

Wage (Rs.)	330	300	470	500	410	380	425	360

Solution: Range = Highest Value − Lowest Value

= 500 – 300 = 200

Coefficient of Range = $\frac{\text{Highest Value} - \text{Lowest Value}}{\text{Highest Value} + \text{Lowest Value}}$

= $\frac{500 - 300}{500 + 300}$ = .25

6.5.2 Range in Discrete Series:

Example 2. Following are the daily wages of workers, find out value of Range and Coefficient of Range.

Wage (Rs.)	300	330	360	380	410	425	470	500
No. of Workers	5	8	12	20	18	15	13	9

Solution: Range = Highest Value − Lowest Value

= 500 – 300 = 200

Coefficient of Range = $\frac{\text{Highest Value} - \text{Lowest Value}}{\text{Highest Value} + \text{Lowest Value}}$

$$= \frac{500 - 300}{500 + 300} \qquad = .25$$

6.5.3 Range in Continuous Series:

Example 3. Following are the daily wages of workers, find out value of Range and Coefficient of Range.

Wage (Rs.)	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90
No. of Workers	5	8	12	20	18	15	13	9

Solution: Range = Highest Value − Lowest Value

= 90 – 10 = 80

$$\text{Coefficient of Range} = \frac{\text{Highest Value} - \text{Lowest Value}}{\text{Highest Value} + \text{Lowest Value}}$$

$$= \frac{90 - 10}{90 + 10} \qquad = .80$$

6.5.4 Merits and Limitations of Range

As far as the merits are concerned,

1. Range is one of the easiest and simplest methods of dispersion.
2. The range is a measure that is rigidly defined.
3. This method gives a broad picture of variation in the data.
4. Range is very useful in various fields of business such as quality control and checking the difference between share prices in the stock exchange.
5. Range is also useful in forecasting.

Limitations of range

1. Range is not an exact measure of depreciation as only gives vague picture.
2. It is not based on all the values of data.
3. It is affected by the extreme values of the data.
4. It is also affected by fluctuations in the sample.
5. In the case of open-ended series range cannot be calculated.

TEST YOUR PROGRESS (A)

1. Compute for the following data Range and Coefficient of Range

28	110	27	77	19	94	63	25	111

2. Given below are heights of students of the two classes. Compare Range of the heights:

Class I	167	162	155	180	182	175	185	158
Class II	169	172	168	165	177	180	195	167

3. Find Range and coefficient of Range

X	5	10	15	20	25	30	35	40
f	6	4	12	7	24	21	53	47

4. Calculate the coefficient of Range:

X:	10-20	20-30	30-40	40-50	50-60
F:	8	10	12	8	4

Answers:

1) 92, 0.7, 2) .088, .083, 3) 35, 0.778, 4) .714

6.6 MEASURE OF DISPERSION – QUARTILE DEVIATION

The range is simple to calculate but suffers from the limitation that it takes into account only extreme values of the data and gives a vague picture of variation. Moreover, it cannot be calculated in case of an open-end series. In such case, we can use another method of Deviation which is Quartile Deviation or Quartile Range. The Quartile Range is the difference between Third Quartile and the First Quartile of the data. Following is a formula for calculating Quartile Range.

$$\textbf{Quartile Range} = Q_3 - Q_1$$

Quartile Deviation: Quartile deviation is the Arithmetic mean of the difference between Third Quartile and the First Quartile of the data.

$$\textbf{Quartile Deviation} = \frac{Q_3 - Q_1}{2}$$

Coefficient of Quartile Deviation: Coefficient of Quartile Deviation is relative measure of Quartile Deviation and can be calculated using the following formula.

$$\textbf{Coefficient of Range} = \frac{Q_3 - Q_1}{Q_3 + Q_1}$$

6.6.1 Quartile Deviation in Individual Series:

Example 4. Following are daily wages of workers, find out value of Quartile Range, Quartile Deviation and Coefficient of Quartile Deviation.

Wage (Rs.)	300	330	380	410	425	470	500

Solution: Q_1 = Value of $\frac{N+1}{4}$ th item = Value of $\frac{7+1}{4}$ th item

= Value of 2n item = 330

Q3 = Value of $\frac{3(N+1)}{4}$ th item = Value of $\frac{3(7+1)}{4}$ th item

= Value of 6th item = 470

Quartile Range = $Q_3 - Q_1$

$= 470 – 330 \qquad = 140$

Quartile Deviation = $\frac{Q_3 - Q_1}{2}$

$= \frac{470 - 330}{2} \qquad = 70$

Coefficient of Quartile Deviation = $\frac{Q_3 - Q_1}{Q_3 + Q_1}$

$= \frac{470 - 330}{470 + 330} \qquad = .175$

6.6.2 Quartile Deviation in Discrete Series:

Example 5. Following are daily wages of workers, find out value of Quartile Range, Quartile Deviation and Coefficient of Quartile Deviation.

Wage (Rs.)	300	330	380	410	425	470	500
No. of Workers	5	8	12	20	18	15	13

Solution: Calculation of Quartile

Wage (Rs.) (X)	**No. of Workers (f)**	**Cumulative Frequency (cf)**
300	5	5
330	8	13
380	12	25
410	20	45
425	18	63
470	15	78
500	13	91

$Q_1 =$ Value of $\frac{N+1}{4}$ th item = Value of $\frac{91+1}{4}$ th item

= Value of 23rd item = 380

Q3 = Value of $\frac{3(N+1)}{4}$ th item = Value of $\frac{3(91+1)}{4}$ th item

= Value of 69th item = 470

Quartile Range = $Q_3 – Q_1$

$= 470 – 380 \qquad = 90$

Quartile Deviation = $\frac{Q_3 - Q_1}{2}$

$= \frac{470 - 380}{2} \qquad = 45$

Coefficient of Quartile Deviation = $\frac{Q_3 - Q_1}{Q_3 + Q_1}$

$= \frac{470 - 380}{470 + 380} \qquad = .106$

6.6.3 Quartile Deviation in Continuous Series:

Example 6. Following are daily wages of workers, find out value of Quartile Range, Quartile Deviation and Coefficient of Quartile Deviation.

Wage (Rs.)	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90
No. of Workers	5	8	12	20	18	15	13	9

Solution: Calculation of Quartile

Wage (Rs.) (X)	**No. of Workers (f)**	**Cumulative Frequency (cf)**
10-20	5	5
20-30	8	13
30-40	12	25
40-50	20	45
50-60	18	63
60-70	15	78
70-80	13	91
80-90	9	100

Calculation of Q_1

Q_1 Class = Value of $\frac{N}{4}$ th item = Value of $\frac{100}{4}$ th item

Q_1 Class = Value of 25th item

Q_1 Class = 30-40

$$Q_1 = L_1 + \frac{\frac{n}{4} - cf}{f} \times c$$

Where $L_1 = 30$, $n = 100$; $cf = 13$; $f = 12$; $c = 10$

$$Q_1 = 30 + \frac{\frac{100}{4} - 13}{12} \times 10 = 40$$

Calculation of Q_3

Q_3 Class = Value of $\frac{3N}{4}$ th item = Value of $\frac{300}{4}$ th item

Q_3 Class = Value of 75th item

Q_3 Class = 60-70

$$Q_3 = L_1 + \frac{\frac{3n}{4} - cf}{f} \times c$$

Where $L_1 = 60$, $n = 100$; $cf = 63$; $f = 15$; $c = 10$

$$Q_1 = 60 + \frac{\frac{3(100)}{4} - 63}{15} \times 10 = 68$$

Calculation of Quartile Range, Quartile Deviation and Coefficient of Quartile Deviation

$$\text{Quartile Range} = Q_3 - Q_1$$

$$= 68 - 40 \qquad = 28$$

$$\text{Quartile Deviation} = \frac{Q_3 - Q_1}{2}$$

$$= \frac{68 - 40}{2} \qquad = 14$$

$$\text{Coefficient of Quartile Deviation} = \frac{Q_3 - Q_1}{Q_3 + Q_1}$$

$$= \frac{68 - 40}{68 + 40} \qquad = .259$$

6.6.4 Merits and Limitations of Quartile Deviation

1. Quartile deviation is a tool which is easy to calculate and understand.
2. Quartile deviation is the best tool of dispersion in case of open-ended series.
3. This method of dispersion is better than range.
4. Unlike the range, it is not affected by the extreme values.
5. This method of dispersion is rigidly defined.
6. This method is very useful specially when we want to know the variability of middle half of the data. Under this method first 25% items that are less than Q_1 and upper 25% items that are more than Q_3 are excluded and only middle 50% items are taken.

Limitations of Quartile Deviation

1. Quartile deviation considers only middle 50% items of the data and ignore rest of the items.
2. It is not possible to make any further algebraic treatment of the quartile deviation.
3. It is not based on all the items.
4. Quartile deviation is highly affected by fluctuation in the sample.
5. It is comparatively difficult to calculate quartile deviation than range.

TEST YOUR PROGRESS (B)

1. Find Quartile deviation and coefficient of Quartile Deviation:

X: 59, 60, 65, 64, 63, 61, 62, 56, 58, 66

2. Find Quartile deviation and coefficient of Quartile Deviation:

X	58	59	60	61	62	63	64	65	66
F	15	20	32	35	33	22	20	10	8

3. Find Quartile deviation and coefficient of Quartile Deviation

X	0-100	100-200	200-300	300-400	400-500	500-600	600-700
F:	8	16	22	30	24	12	6

4. Calculate Inter Quartile Range, Q.D and coefficient of Q.D

X	0-10	10-20	20-30	30-40	0-500	50-60	60-70	70-80	80-90
F:	11	18	25	28	30	33	22	15	22

Answers

1. 2.75, 0.0447, 2. 1.5, .024
3. 113.54, 0.335, 4. 34.84, 17.42, .3769

6.7 MEASURE OF DISPERSION – MEAN DEVIATION

Both Range and Quartile Deviation are positional methods of Dispersion and take into consideration only two values. Range considers only highest and lowest values while calculating Dispersion, while Quartile Deviation considers on First and Third Quartile for calculating Dispersion. Both these methods are not based on all the values of the data and are considerably affected by the sample unit. A good measure of Dispersion considers all the values of data.

Mean Deviation is a tool for measuring the Dispersion that is based on all the values of Data. Contrary to its name, it is not necessary to calculate Mean Deviation from Mean, it can also be calculated using the Median of the data or Mode of the data. In the Mean deviation, we calculated deviations of the items of data from its Average (Mean, Median or Mode) by taking positive signs only. When we divide the sum of deviation by the number of items, we get the value of Mean Deviation. In simple words:

"Mean Deviation is the value obtained by taking the arithmetic mean of the deviations obtained by deducting average of data whether Mean, Median or Mode from values of data, ignoring the signs of the deviations."

6.7.1 Mean Deviation in case of Individual Series:

As we have already discussed that Mean Deviation can be calculated from Mean, Median or Mode. Following are the formula for calculating Mean Deviation in case of Individual series.

$$\textbf{Mean Deviation from Mean } (MD_{\bar{X}}) = \frac{\sum |X - \bar{X}|}{n} = \frac{\sum |D_{\bar{X}}|}{n}$$

$$\textbf{Mean Deviation from Median } (MD_{M}) = \frac{\sum |X - M|}{n} = \frac{\sum |D_{M}|}{n}$$

$$\textbf{Mean Deviation from Mode } (MD_Z) = \frac{\sum |X - Z|}{n} = \frac{\sum |D_Z|}{n}$$

In case we want to calculate Coefficient of Mean Deviation, it can be done using the following formulas.

$$\textbf{Coefficient of Mean Deviation from Mean } (MD_{\bar{X}}) = \frac{MD_{\bar{X}}}{\bar{X}}$$

$$\textbf{Coefficient of Mean Deviation from Median } (MD_M) = \frac{MD_M}{M}$$

$$\textbf{Coefficient of Mean Deviation from Mode } (MD_Z) = \frac{MD_Z}{Z}$$

Example 7. Following are the marks obtained by Students of a class in a test. Calculated Mean Deviation from (i) Mean (ii) Median (iii) Mode. Also calculate Coefficient of Mean Deviation.

Wage (Rs.)	5	7	8	8	9	11	13	14	15

Solution: Let us calculate Mean Median and Mode

Mean $(\bar{X}) = \frac{5+7+8+8+9+11+13+14+15}{9} = \frac{90}{9} = 10$

Median (M) = Value of $\frac{N+1}{2}$ th item = Value of $\frac{9+1}{2}$ th item

= Value of 5th item = 9

Mode = Item having maximum frequency i.e. 8.

Calculation of Deviations

Marks X	$D_{\bar{X}} = \lvert X - \bar{X} \rvert$ (Where $\bar{X}$ = 10)	$D_M = \lvert X - M \rvert$ (Where M = 9)	$D_Z = \lvert X - Z \rvert$ (Where Z = 8)
5	5	4	3
7	3	2	1
8	2	1	0
8	2	1	0
9	1	0	1
11	1	2	3
13	3	4	5
14	4	5	6
15	5	6	7
	$\sum D_{\bar{X}} = 26$	$\sum D_M = 25$	$\sum D_Z = 26$

1. Mean Deviation from Mean $(MD_{\bar{X}}) = \frac{\sum |X - \bar{X}|}{n} = \frac{\sum |D_{\bar{X}}|}{n} = \frac{26}{9} = 2.88$

Coefficient of Mean Deviation from Mean ($MD_{\bar{X}}$) = $\frac{MD_{\bar{X}}}{\bar{X}} = \frac{2.88}{10}$ = .288

2. Mean Deviation from Median (MD_M) = $\frac{\sum|X-M|}{n} = \frac{\sum|D_M|}{n} = \frac{25}{9}$ = 2.78

Coefficient of Mean Deviation from Median (MD_M) = $\frac{MD_M|}{M} = \frac{2.78}{9}$ = .309

3. Mean Deviation from Mode (MD_Z) = $\frac{\sum|X-Z|}{n} = \frac{\sum|D_Z|}{n} = \frac{26}{9}$ = 2.88

Coefficient of Mean Deviation from Mode (MD_Z) = $\frac{MD_Z}{Z} = \frac{2.88}{8}$ = .36

6.7.2 Mean Deviation in case of Discrete Series:

Following is the formula for calculating Mean Deviation in case of Discrete series.

$$\textbf{Mean Deviation from Mean } (MD_{\bar{X}}) = \frac{\sum f|X-\bar{X}|}{n} = \frac{\sum f|D_{\bar{X}}|}{n}$$

$$\textbf{Mean Deviation from Median } (MD_M) = \frac{\sum f|X-M|}{n} = \frac{\sum f|D_M|}{n}$$

$$\textbf{Mean Deviation from Mode } (MD_Z) = \frac{\sum f|X-Z|}{n} = \frac{\sum f|D_Z|}{n}$$

Example 8. Following are the wages of workers that are employed in a factory. Calculate Mean Deviation from (i) Mean (ii) Median (iii) Mode. Also, calculate Coefficient of Mean Deviation.

Wage (Rs.)	300	330	380	410	425	470	500
No. of Workers	6	8	15	25	18	15	13

Solution: Let us calculate Mean Median and Mode

X	f	fX	cf
300	5	1500	5
330	8	2640	13
380	15	5700	28
410	26	10660	54
425	18	7650	72
470	15	7050	87
500	13	6500	100
		∑X = 41700	

Mean ($\bar{X}$) = $\frac{\sum X}{n} = \frac{41700}{100}$ = 417

Median (M) = Value of $\frac{N+1}{2}$ th item = Value of $\frac{100+1}{2}$ th item

= Value of 50.5 item = 410

Mode = Item having maximum frequency i.e. 410.

Calculation of Deviations

X	f	$D_{\bar{X}} = \lvert X - \bar{X} \rvert$ ($\bar{X}$ = 417)	$fD_{\bar{X}}$	$D_M = \lvert X - M \rvert$ (M = 410)	fD_M	$D_Z = \lvert X - Z \rvert$ (Z = 410)	fD_Z
300	5	117	585	110	550	110	550
330	8	87	696	80	640	80	640
380	15	37	555	30	450	30	450
410	26	7	182	0	0	0	0
425	18	8	144	15	270	15	270
470	15	53	795	60	900	60	900
500	13	83	1079	90	1170	90	1170
			$\sum fD_{\bar{X}}$ = 4036		$\sum fD_M$ = 3980	$\sum D_Z$ = 26	$\sum fD_Z$ = 3980

1. Mean Deviation from Mean ($MD_{\bar{X}}$) = $\frac{\sum f\lvert X - \bar{X}\rvert}{n} = \frac{\sum f\lvert D_{\bar{X}}\rvert}{n} = \frac{4036}{100}$ = 40.36

Coefficient of Mean Deviation from Mean ($MD_{\bar{X}}$) = $\frac{MD_{\bar{X}}}{\bar{X}} = \frac{40.36}{417}$ = .097

2. Mean Deviation from Median (MD_M) = $\frac{\sum f\lvert X - M\rvert}{n} = \frac{\sum f\lvert D_M\rvert}{n} = \frac{3980}{100}$ = 39.80

Coefficient of Mean Deviation from Median (MD_M) = $\frac{MD_M\rvert}{M} = \frac{39.80}{410}$ = .097

3. Mean Deviation from Mode (MD_Z) = $\frac{\sum f\lvert X - Z\rvert}{n} = \frac{\sum f\lvert D_Z\rvert}{n} = \frac{3980}{100}$ = 39.80

Coefficient of Mean Deviation from Mode (MD_Z) = $\frac{MD_Z}{Z} = \frac{39.80}{410}$ = .097

6.7.3 Mean Deviation in case of Continuous Series:

In case of calculation of Mean Deviation in continuous series, the formula will remain same as we have done in Discrete Series but only difference is that instead of taking deviation from Data, we take deviations from mid value of the data. Further in case of continuous series also the Mean Deviation can be calculated from Mean, Median or Mode. However, in most of the cases it is calculated from Median. Following formulas are used for continuous series:

Mean Deviation from Mean ($MD_{\bar{X}}$) = $\frac{\sum f\lvert X - \bar{X}\rvert}{n} = \frac{\sum f\lvert D_{\bar{X}}\rvert}{n}$

Mean Deviation from Median (MD_M) = $\frac{\sum f\lvert X - M\rvert}{n} = \frac{\sum f\lvert D_M\rvert}{n}$

Mean Deviation from Mode (MD_Z) = $\frac{\sum f\lvert X - Z\rvert}{n} = \frac{\sum f\lvert D_Z\rvert}{n}$

Example 9. Following are daily wages of workers, find out value of Mean Deviation and Coefficient of Mean Deviation.

Wage (Rs.)	10-20	20-30	30-40	40-50	50-60	60-70	70-80	80-90
No. of Workers	5	8	12	20	18	15	13	9

Solution:

Wage (Rs.) (X)	**No. of Workers (f)**	**Cumulative Frequency (cf)**	**Mid Value m**	**$\vert D_M \vert$ $\vert m - M \vert$**	**$\vert f D_M \vert$**
10-20	5	5	15	37.78	188.9
20-30	8	13	25	27.78	222.24
30-40	12	25	35	17.78	213.36
40-50	20	45	45	7.78	155.6
50-60	18	63	55	2.22	39.96
60-70	15	78	65	12.22	183.3
70-80	13	91	75	22.22	288.86
80-90	9	100	85	32.22	289.98
	N = 100				**$\sum \vert f D_M \vert = 1582.2$**

Calculation of Median

$$\text{Median Class} = \text{Value of } \frac{N}{2} \text{ th item} = \text{Value of } \frac{100}{2} \text{ th item}$$

Median Class = Value of 50th item

Median Class = 50-60

$$M = L_1 + \frac{\frac{n}{2} - cf}{f} \times c$$

Where $L_1 = 50$, $n = 100$; $cf = 45$; $f = 18$; $c = 10$

$$M = 50 + \frac{\frac{100}{2} - 45}{18} \times 10 = 52.78$$

Calculation of Mean Deviation from Median

$$\text{Mean Deviation from Median } (MD_M) = \frac{\sum f| X - M|}{n} = \frac{\sum f| D_M|}{n} = \frac{1582.2}{100} = 15.82$$

$$\text{Coefficient of Mean Deviation from Median } (MD_M) = \frac{MD_M|}{M} = \frac{15.82}{52.78} = .30$$

6.7.4 Merits and Limitations of Mean Deviation

1. We can calculate mean deviation very easily.
2. Mean deviation is based on all the items of the Data. Change in any value of the data is also going to affect mean deviation.
3. As it is based on all the items of the data, it is not affected by the extreme values of the data.

4. Mean deviation can be calculated from Mean, Median or Mode.
5. Mean deviation is a rigidly defined method of measuring dispersion.
6. Mean deviation can be used for comparison of two different series.

Limitations of Mean Deviation

1. While calculating the mean deviation, we consider only positive sign and ignore the negative sign.
2. In case mean deviation is calculated from mode, it is not a reliable measure of dispersion as mode is not a true representative of the series.
3. It is very difficult to calculate Mean Deviation in case of open-ended series.
4. Mean deviation is not much capable of further statistical calculations.
5. In case we have Mean Deviation of two different series, we cannot calculate combined mean deviation of the data.
6. In case value of Mean, Median or Mode is in fraction, it is difficult to calculate mean deviation.

TEST YOUR PROGRESS (C)

1. Calculate Mean Deviation from i) Mean, ii) Median, iii) Mode

X: 7, 4, 10, 9, 15, 12, 7, 9, 7

2. With Median as base calculate Mean Deviation of two series and compare variability:

Series A:	3484	4572	4124	3682	5624	4388	3680	4308
Series B:	487	508	620	382	408	266	186	218

3. Calculate Co-efficient of mean deviation from Mean, Median and Mode from the following data

X:	4	6	8	10	12	14	16
f:	2	1	3	6	4	3	1

4. Calculate Co-efficient of Mean Deviation from Median.

X;	20-25	25-30	30-40	40-45	45-50	50-55	55-60	60-70	70-80
F:	7	13	16	28	12	9	7	6	2

5. Calculate M.D. from Mean and Median

X	0-10	10-20	20-30	30-40	40-50
f	6	28	51	11	4

6. Calculate Co-efficient of Mean Deviation from Median.

X; 16-20 21-25 26-30 31-35 36-40 41-45 46-50 51-55 56-60

F:	8	13	15	20	11	7	3	2	1

Answers

1. 2.35, 2.33, 2.56, 2. 11.6%, 30.73%, 3. 0.239, 0.24, 0.24, 4. 0.214

5. M.D. (Mean) =6.572, Coefficient of M.D. (Mean) =0.287, M.D. (Median) =6.4952, Coefficient of M.D. (Median) 0.281, 6. 0.22

6.8 MEASURE OF DISPERSION – STANDARD DEVIATION

Standard deviation is assumed as best method of calculating deviations. This method was given by great statistician Karl Pearson in the year 1893. In case of Mean deviation, when we take deviations from the actual mean, the sum of deviations is always zero. To avoid this problem, we have to ignore the sign of the deviations. However, in case of Standard Deviation this problem is solved by taking the square of the deviations because when we take a square of the negative sign, it is also converted into the positive sign. Then after calculating the Arithmetic mean of the deviations, we again take square root, to find out standard deviation. In other words, we can say that "Standard Deviation is the square root of the Arithmetic mean of the squares of deviation of the item from its Arithmetic mean."

The standard deviation is always calculated from the Arithmetic mean and is an absolute measure of finding the dispersion. We could also find a relative measure of standard deviation which is known as coefficient of standard deviation.

Coefficient of Standard Deviation – Coefficient of Deviation is the relative measure of the standard deviation and can be calculated by dividing the Value of Standard Deviation with the Arithmetic Mean. The value of coefficient always lies between 0 and 1, where 0 indicates no Standard Deviation and 1 indicated 100% standard deviation. Following is the formula for calculating coefficient of Standard Deviation.

$$\textbf{Coefficient of Standard Deviation} = \frac{SD}{\bar{X}}$$

Coefficient of Variation – Coefficient of Variation is also relative measure of the standard deviation, but unlike Coefficient of Standard Deviation it is not represented in fraction rather it is represented in terms of % age. It can be calculated by dividing the Value of Standard Deviation with the Arithmetic Mean and then multiplying resulting figure with 100. The value of coefficient always lies between 0 and 100. Following is the formula for calculating coefficient of Standard Deviation. Low Coefficient of Variation implies less variation, more uniformity and reliability.

Contrary to this higher Coefficient of Variation implies more variation, less uniformity and reliability.

$$\textbf{Coefficient of Standard Deviation} = \frac{SD}{\bar{X}} \times 100$$

Variance – Variance is the square of the Standard Deviation. In other words, it is Arithmetic mean of square of Deviations taken from Actual Mean of the data. This term was first time used by R. A. Fischer in 1913. He used Variance in analysis of financial models. Mathematically:

$$\textbf{Variance} = (\textbf{Standard Deviation})^2 \text{ or } \sigma^2$$

6.8.1 Standard Deviation in case of Individual Series

Following are the formula for calculating Standard Deviation in case of the Individual Series:

1. **Actual Mean Method –** In this method we take deviations from actual mean of the data.

   $$\textbf{Standard Deviation (SD or } \boldsymbol{\sigma}\textbf{)} = \sqrt{\frac{\sum x^2}{n}}$$

   Where $x = X - \bar{X}$, n = Number of Items.
2. **Assumed Mean Method -** In this method we take deviations from assumed mean of the data. Any number can be taken as assumed mean, however for sake of simplicity it is better to take whole number as assumed mean.

   $$\textbf{Standard Deviation (SD or } \boldsymbol{\sigma}\textbf{)} = \sqrt{\frac{\sum dx^2}{n} - \left(\frac{\sum dx}{n}\right)^2}$$

   Where dx = X – A, n = Number of Items.

3. **Direct Methods -** In this method we don't take deviations and standard deviation is calculated directly from the data.

   $$\textbf{Standard Deviation (SD or } \boldsymbol{\sigma}\textbf{)} = \sqrt{\frac{\sum X^2}{n} - \left(\frac{\sum X}{n}\right)^2}$$

Example 10. Following are the marks obtained by Students of a class in a test. Calculate Standard Deviation using (i) Actual Mean (ii) Assumed Mean (iii) Direct Method. Also calculate Coefficient of Standard Deviation.

Marks	5	7	11	16	15	12	18	12

Solution:

1. Standard Deviation using Actual Mean

Marks X	**$x = X - \bar{X}$ (Where $\bar{X} = 12$)**	**x^2**
5	-7	49
7	-5	25

11	-1	01
16	4	16
15	3	09
12	0	00
18	6	36
12	0	00
$\sum X = 96$		**$\sum x^2 = 136$**

Mean ($\overline{X}$) = $\frac{\sum X}{n} = \frac{96}{8} = 12$

Standard Deviation (SD or **σ**) = $\sqrt{\frac{\sum x^2}{n}} = \sqrt{\frac{136}{8}} = \sqrt{17}$ =4.12

Coefficient of Standard Deviation = $\frac{SD}{\overline{X}} = \frac{4.12}{12}$ = .34

2. Standard Deviation using Assumed Mean

Marks X	**dx = X - A (Where A = 11)**	**dx^2**
5	-6	36
7	-4	16
11	0	00
16	5	25
15	4	16
12	1	01
18	7	49
12	1	01
$\sum X = 96$	**$\sum dx = 8$**	**$\sum dx^2 = 144$**

Mean ($\overline{X}$) =A+ $\frac{\sum dX}{n} = 11 + \frac{8}{8} = 12$

Standard Deviation (**σ**) = $\sqrt{\frac{\sum dx^2}{n} - \left(\frac{\sum dx}{n}\right)^2} = \sqrt{\frac{144}{8} - \left(\frac{8}{8}\right)^2} = \sqrt{18-1} = \sqrt{17} = 4.12$

Coefficient of Standard Deviation = $\frac{SD}{\overline{X}} = \frac{4.12}{12}$ = .34

3. Standard Deviation by Direct Method

Marks X	**X^2**
5	25
7	49
11	121
16	256
15	225
12	144

18 12	324 144
∑X = 96	**∑X2 = 1288**

Mean ($\bar{X}$) = $\frac{\sum X}{n} = \frac{96}{8}$ = 12

Standard Deviation (**σ**) = $\sqrt{\frac{\sum X^2}{n} - \left(\frac{\sum X}{n}\right)^2} = \sqrt{\frac{1288}{8} - \left(\frac{96}{8}\right)^2} = \sqrt{161 - 144} = \sqrt{17} = 4.12$

Coefficient of Standard Deviation = $\frac{SD}{\bar{X}} = \frac{4.12}{12}$ = .34

Example 11. Two Players scored following scores in 10 cricket matches. On base of their performance find out which is better scorer and also find out which player is more consistent.

Player X	26	24	28	30	35	40	25	30	45	17
Player Y	10	15	24	26	34	45	25	31	20	40

Solution: Mean and Standard Deviation of Player X

Score **X**	**x = X - $\bar{X}$** **(Where $\bar{X}$ = 30)**	**x^2**
26 24 28 30 35 40 25 30 45 17	-4 -6 -2 0 5 10 -5 0 15 -13	16 36 2 0 25 100 25 0 225 169
∑X = 300		**$\sum x^2$ = 600**

Mean ($\bar{X}$) = $\frac{\sum X}{n} = \frac{300}{10}$ = 30

Standard Deviation (SD or **σ**) = $\sqrt{\frac{\sum x^2}{n}} = \sqrt{\frac{600}{10}} = \sqrt{60}$ = 7.746

Coefficient of Variation = $\frac{SD}{\bar{X}}$ × 100= $\frac{7.746}{30}$ × 100 = 25.82%

Mean and Standard Deviation of Player Y

Score **Y**	**y = Y - $\bar{Y}$** **(Where $\bar{Y}$ = 27)**	**y^2**
10 15	-17 -12	289 144

24	-3	9
26	-1	1
34	7	49
45	18	324
25	-2	4
31	4	16
20	-7	49
40	13	169
$\sum X$ = 270		**$\sum x^2$ = 1054**

Mean ($\overline{Y}$) = $\frac{\sum Y}{n} = \frac{270}{10}$ = 27

Standard Deviation (SD or **σ**) = $\sqrt{\frac{\sum y^2}{n}} = \sqrt{\frac{1054}{10}} = \sqrt{105.40}$ = 10.27

Coefficient of Variation = $\frac{SD}{\overline{Y}}$ × 100= $\frac{10.27}{27}$ × 100 = 38.02%

Conclusion:

1. As average score of Player X is more than Player Y, he is better scorer.
2. As Coefficient of Variation of Player X is less than Player Y, he is more consistent also.

6.8.2 Standard Deviation in case of Discrete Series

Following are the formula for calculating Standard Deviation in case of the Discrete Series:

1. **Actual Mean Method –** In this method we take deviations from actual mean of the data.

   Standard Deviation (SD or σ) = $\sqrt{\frac{\sum fx^2}{n}}$

   Where x = X - $\overline{X}$, f = Frequency, n = Number of Items.

2. **Assumed Mean Method -** In this method we take deviations from assumed mean of the data.

   Standard Deviation (SD or σ) = $\sqrt{\frac{\sum fdx^2}{n} - \left(\frac{\sum fdx}{n}\right)^2}$

   Where dx = X – A, n = Number of Items.

3. **Direct Methods -** In this method we don't take deviations and standard deviation is calculated directly from the data.

   Standard Deviation (SD or σ) = $\sqrt{\frac{\sum fX^2}{n} - \left(\frac{\sum fX}{n}\right)^2}$

Example 12. Following are the marks obtained by Students of a class in a test. Calculate Standard Deviation using (i) Actual Mean (ii) Assumed Mean (iii) Direct Method.

Marks	5	10	15	20	25	30	35
Frequency	2	7	11	15	10	4	1

Solution: **1. Standard Deviation using Actual Mean**

Marks X	f	fX	$x = X - \bar{X}$ ($\bar{X} = 19$)	x^2	fx^2
5	2	10	-14	196	392
10	7	70	-9	81	567
15	11	165	-4	16	176
20	15	300	1	1	15
25	10	250	6	36	360
30	4	120	11	121	484
35	1	35	16	256	256
	N = 50	**$\sum fX = 950$**			**$\sum x^2 = 2250$**

Mean ($\bar{X}$) = $\frac{\sum fX}{n} = \frac{950}{50} = 19$

Standard Deviation (SD or **σ**) = $\sqrt{\frac{\sum fx^2}{n}} = \sqrt{\frac{2250}{50}} = \sqrt{45} = 6.708$

2. Standard Deviation using Assumed Mean

Marks X	f	dx = X - A (A = 20)	dx^2	fdx	fdx^2
5	2	-15	225	-30	450
10	7	-10	100	-70	700
15	11	-5	25	-55	275
20	15	0	0	0	0
25	10	5	25	50	250
30	4	10	100	40	400
35	1	15	225	15	225
	N = 50			**$\sum fdx = -50$**	**$\sum fdx^2 = 2300$**

Standard Deviation (**σ**) $= \sqrt{\frac{\sum fdx^2}{n} - \left(\frac{\sum fdx}{n}\right)^2}$

$= \sqrt{\frac{2300}{50} - \left(\frac{-50}{50}\right)^2} = \sqrt{46 - 1} = \sqrt{45} = 6.708$

3. Standard Deviation using Direct Method

Marks (X)	f	X^2	fX	fX^2

5 10 15 20 25 30 35	2 7 11 15 10 4 1	25 70 225 400 625 900 1225	10 70 165 300 250 120 35	125 700 2475 6000 6250 3600 1225
	N = 50		**$\sum fX = 950$**	**$\sum fX^2 = 20300$**

Standard Deviation (**σ**) $= \sqrt{\frac{\sum fX^2}{n} - \left(\frac{\sum fX}{n}\right)^2}$

$= \sqrt{\frac{20300}{50} - \left(\frac{950}{50}\right)^2} = \sqrt{406 - 361} = \sqrt{45} = 6.708$

6.8.3 Standard Deviation in case of Continuous Series

In case of continuous series, the calculation will remain same as in case of discrete series but the only difference is that instead of taking deviations from data, deviations are taken from Mid value of the data. Formulas are same as discussed above for discrete series.

Example 13. Following are the marks obtained by Students of a class in a test. Calculate Standard Deviation using (i) Actual Mean (ii) Assumed Mean (iii) Direct Method. Also calculate coefficient of variation and Variance.

Marks	5-10	10-15	15-20	20-25	25-30	30-35	35-40
Frequency	2	7	11	15	10	4	1

Solution: 1. Standard Deviation using Actual Mean

Marks X	**m**	**f**	**fX**	**$x = m - \bar{X}$ ($\bar{X} = 21.5$)**	**x^2**	**fx^2**
5-10	7.5	2	15	-14	196	392
10-15	12.5	7	87.5	-9	81	567
15-20	17.5	11	192.5	-4	16	176
20-25	22.5	15	337.5	1	1	15
25-30	27.5	10	275	6	36	360
30-35	32.5	4	130	11	121	484
35-40	37.5	1	37.5	16	256	256
		N = 50	**$\sum fX = 1075$**			**$\sum x^2 = 2250$**

Mean ($\bar{X}$) $= \frac{\sum fX}{n} = \frac{1075}{50} = 21.5$

Standard Deviation (SD or **σ**) $= \sqrt{\frac{\sum fx^2}{n}} = \sqrt{\frac{2250}{50}} = \sqrt{45} = 6.708$

2. Standard Deviation using Assumed Mean

Marks X	m	f	dx = X - A (A = 22.5)	dx^2	fdx	fdx^2
5-10	7.5	2	-15	225	-30	450
10-15	12.5	7	-10	100	-70	700
15-20	17.5	11	-5	25	-55	275
20-25	22.5	15	0	0	0	0
25-30	27.5	10	5	25	50	250
30-35	32.5	4	10	100	40	400
35-40	37.5	1	15	225	15	225
		N = 50			**$\sum$fdx = -50**	**$\sum fdx^2$ = 2300**

Standard Deviation (**σ**) $=\sqrt{\frac{\sum fdx^2}{n} - \left(\frac{\sum fdx}{n}\right)^2}$

$=\sqrt{\frac{2300}{50} - \left(\frac{-50}{50}\right)^2} = \sqrt{46-1} = \sqrt{45} = 6.708$

3. Standard Deviation using Direct Method

Marks X	m	f	X^2	fX	fX^2
5-10	7.5	2	56.25	15	112.5
10-15	12.5	7	156.25	87.5	1093.75
15-20	17.5	11	306.25	192.5	3368.75
20-25	22.5	15	506.25	337.5	7593.75
25-30	27.5	10	756.25	275	7562.5
30-35	32.5	4	1056.25	130	4225
35-40	37.5	1	1406.25	37.5	1406.25
		N = 50		**$\sum$fX = 1075**	**$\sum fX^2$ = 25366.5**

Standard Deviation (**σ**) $=\sqrt{\frac{\sum fX^2}{n} - \left(\frac{\sum fX}{n}\right)^2} = \sqrt{\frac{25366.5}{50} - \left(\frac{1075}{50}\right)^2} = \sqrt{507.25 - 462.25}$

$= \sqrt{45} = 6.708$

Coefficient of Standard Deviation = $\frac{SD}{\bar{X}} \times 100 = \frac{6.708}{21.5} \times 100 = 31.2\%$

Variance =(Standard Deviation)^2 or $\sigma^2 = (6.708)^2 = 45$

6.8.4 Combined Standard Deviation

The main benefit of standard deviation is that if we know the mean and standard deviation of two or more series, we can calculate combined standard deviation of all the series. This feature is not available in other measures of dispersion. That’s why we assume that standard deviation is best measure of finding the dispersion. Following formula is used for this purpose:

$$\sigma_{123} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_3\sigma_3^2 + n_1 d_1^2 + n_2 d_2^2 + n_3 d_3^2}{n_1 + n_2 + n_3}}$$

Where, n_1, n_2, n_3 = number of items in series 1, 2 and 3

σ_1, σ_2, σ_3 = standard deviation of series 1, 2 and 3

d_1, d_2, d_3 = difference between mean of the series and combined mean for 1, 2 and 3.

Example14. Find the combined standard deviation for the following data

	Firm A	**Firm B**
No. of Wage Workers	**70**	**60**
Average Daily Wage (Rs.)	**40**	**35**
S.D of wages	**8**	**10**

Solution: Combined mean wage of all the workers in the two firms will be

$$\overline{X_{12}} = \frac{N_1\overline{X_1} + N_2\overline{X_2}}{N_1 + N_2}$$

Where N_1 = Number of workers in Firm A

N_2 = Number of workers in Firm B

$\overline{X_1}$ = Mean wage of workers in Firm A

and $\overline{X_2}$ = Mean wage of workers in Firm B

We are given that

$N_1 = 70$ $\quad N_2 = 60$

$\overline{X_1} = 40$ $\quad \overline{X_2} = 35$

∴ Combined Mean, $\overline{X_{12}}$

$$= \frac{(70\times40)+(60\times35)}{70+60}$$

$$= \frac{4900}{130}$$

$$= \text{Rs. } 37.69$$

Combined Standard Deviation = $\sigma_{123} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_1 d_1^2 + n_2 d_2^2}{n_1 + n_2}}$

d_1 = 40 – 37.69 = 2.31

d_2 = 35 – 37.69 = -2.69

$$\sigma_{123} = \sqrt{\frac{70\,(8)^2 + 60\,(10)^2 + 70\,(2.31)^2 + 60\,(-2.69)^2}{70 + 60}} = 9.318$$

Example 15. Find the missing values

	Firm A	Firm B	Firm C	Combined
No. of Wage Workers	**50**	**?**	**90**	**200**
Average Daily Wage (Rs.)	**113**	?	**115**	**116**
S.D of wages	**6**	**7**	**?**	**7.746**

Solution: Combined n = $n_1 + n_2 + n_3$

$200 = 50 + n_2 + 90$, $N_2 = 60$

Now Combined mean wage of all the workers in the two firms will be

$$\overline{X_{12}} = \frac{N_1\overline{X_1}+N_2\overline{X_2}+N_3\overline{X_3}}{N_1+N_2+N_3}$$

We are given that

$N_1 = 50, N_2 = 60, N_3 = 90, \overline{X_1} = 113, \overline{X_2} = ?, \quad \overline{X_3} = 115, \overline{X_{123}} = 116$

∴ Combined Mean, $\overline{X_{12}}$

$$116 = \frac{(50\times113)+(60\times\overline{X_2})+(90\times115)}{50+60+90}$$

$$116 = \frac{565+(60\times\overline{X_2})+1035}{50+60+90}$$

$2320 = 1600 + 6\,\overline{X_2}$

$\overline{X_2} = 120$

Combined Standard Deviation = $\sigma_{123} = \sqrt{\frac{n_1\sigma_1^2 + n_2\sigma_2^2 + n_3\sigma_3^2 + n_1 d_1^2 + n_2 d_2^2 + n_3 d_3^2}{n_1+n_2+n_3}}$

$d_1 = 113 - 116 = -3$, $d_2 = 120 - 116 = 4$, $d_3 = 115 - 116 = -1$

$\sigma_{123} = \sqrt{\frac{50(6)^2+60(7)^2+90(\sigma_3)^2+50(-3)^2+60(4)^2+90(-1)^2}{50+60+90}} = 7.746$

Squaring both sides

$60 = \frac{180+294+9\sigma_3^2 45+96+9}{200}$

$1200 = 9\,\sigma_3^2 + 624$

$\sigma_3 = 8$

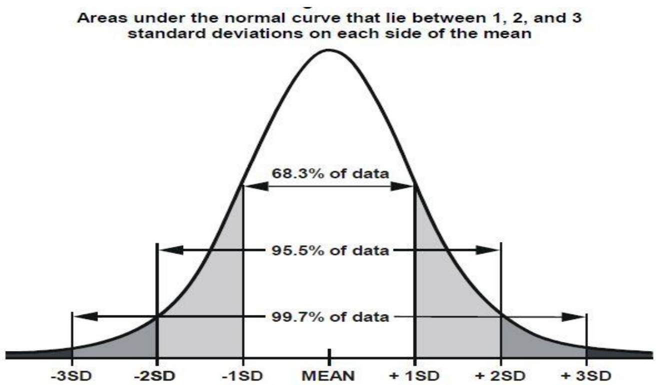
6.8.5 Properties of Standard Deviation

1. Standard Deviation of first 'n' natural numbers is $\sqrt{\frac{n^2-1}{12}}$.

2. It is independent of change in origin it means it is not affected even if some constant is added or subtracted from all the values of the data.
3. It is not independent of change in scale. So, if we divide or multiply all the values of the data with some constant, Standard Deviation is also multiplied or divided by the same constant.
4. We can calculate combined Standard Deviation by following the formula:

$$\sigma_{123} = \sqrt{\frac{n_1 \sigma_1^2 + n_2 \sigma_2^2 + n_3 \sigma_3^2 + n_1 d_1^2 + n_2 d_2^2 + n_3 d_3^2}{n_1 + n_2 + n_3}}$$

5. In case of normal distribution following results are found:



68.27% item lies within the range of: $\overline{X} \pm \sigma$

95.45% item lies within the range of: $\overline{X} \pm 2\sigma$

99.73% item lies within the range of: $\overline{X} \pm 3\sigma$

6. In case of normal distribution there is relation between Quartile Deviation, Mean Deviation and Standard Deviation which is as follows:

   6 (Q.D.) = 5 (M.D.) = 4 (S.D.)

7. In perfect symmetric distribution following result follows:

   Range = 6 (S.D.)

8. When we take square of Standard Deviation it is called Variance.

   Variance = $(S.D)^2$

6.8.6 Merits and Limitations of Standard Deviation

1. It is rigidly defined.
2. It is the best measure to find out deviations.
3. It is based on arithmetic mean.
4. It is based on all the values.

5. We can find the combined standard deviation of different series under this.
6. It is capable of further algebraic treatment.
7. By finding the coefficient of variation, we can compare two different series.

Limitations of Standard Deviation

1. It is comparatively difficult to calculate.
2. It is mostly affected by extreme values.
3. Common people are not aware of the concept of standard deviation.

6.9 MEANING AND MEASURES OF SKEWNESS

Skewness is a statistical measure that helps to assess the asymmetry or lack of symmetry in a probability distribution of a random variable. It indicates the degree to which the values in a dataset are skewed or deviate from a symmetric distribution. Skewness can take positive or negative values or even zero, each indicating a different type of skewness:

a. **Positive Skewness**: If the distribution has a long tail on the right side and the majority of the data is concentrated on the left side, it is said to have positive skewness. The right tail is stretched out, and the mean is typically greater than the median.
b. **Negative Skewness**: If the distribution has a long tail on the left side and the majority of the data is concentrated on the right side, it is said to have negative skewness. The left tail is stretched out, and the mean is typically smaller than the median.
c. **Zero Skewness:** If the distribution is perfectly symmetrical, it has zero skewness. This means that the data is equally distributed on both sides of the mean, and the mean and median are equal.

There are various measures of skewness used to quantify the extent of skewness in a dataset. Some common measures include:

Pearson's First Coefficient of Skewness (moment skewness): It is defined as the third standardized moment of a distribution. The formula for Pearson's first coefficient of skewness is:

Skewness = (3 * (Mean - Median)) / Standard Deviation

Here, Mean refers to the arithmetic mean, Median is the median of the data, and Standard Deviation is the standard deviation of the dataset. Bowley's Skewness Coefficient: It is a measure of skewness based on quartiles. The formula for Bowley's skewness coefficient is:

Skewness = (Q1 + Q3 - 2 * Median) / (Q3 - Q1)

Here, Q1 and Q3 are the first and third quartiles, respectively.

Sample Skewness: It is a measure of skewness based on moments. The formula for sample skewness is:

Skewness = (1 / n) * ∑[(xi - Mean) / Standard Deviation]^3

Here, n is the sample size, xi represents each observation in the dataset, Mean is the arithmetic mean, and Standard Deviation is the standard deviation.

These are some commonly used measures of skewness, and each provides a different perspective on the skewness of the data. It's important to consider multiple measures and examine the data distribution to gain a comprehensive understanding of skewness.

TEST YOUR PROGRESS (D)

1. Calculate Standard Deviation and find Variance:

X:	5	7	11	16	15	12	18	12

2. Two Batsmen X and Y score following runs in ten matches. Find who is better Scorer and who is more consistent.

X:	26	24	28	30	35	40	25	30	45	17
Y:	10	15	24	26	34	45	25	31	20	40

3. Calculate S.D, coefficient of SD, coefficient of Variation:

X	15	25	35	45	55	65
f	2	4	8	20	12	4

4. Find Standard Deviation.

X:	5-10	10-15	15-20	20-25	25-30	30-35
F:	2	9	29	24	11	6

5. Find combined Mean and Combined Standard Deviation:

Part	No. of Items	Mean	S.D.
1	200	25	3
2	250	10	4
3	300	15	5

Answers: 1)4.12, 16.97, 2) X is better and consistent, X means 30 CV 25.82%, Y means 27 CV 38.02%, 3) 11.83, 0.265, 26.5%, 4) 5.74 5) 16, 7.2,

6.10 SUM UP

- Dispersion shows whether the average is a good representative of the series or not.
- High dispersion means values differ more than their average.
- There are two measures of dispersion, Absolute measure and relative measure.
- four methods can be used for measuring the dispersion namely, Range, Quartile Deviation, Mean Deviation and Dispersion.
- Mean deviation can be calculated from Mean, Median or Mode
- Standard Deviation is the best measure of Dispersion.
- If we know the standard deviation of two series, we can calculate the combined standard deviation.

6.11 KEY TERMS

- **Dispersion:** Dispersion shows the extent to which individual items in the data differs from its average. It is a measure of the difference between data and the individual items. It indicates that how that are lacks the uniformity.
- **Range:** Range is the difference between highest value of the data and the lowest value of the data. The more is the difference between highest and the lowest value, more is the value of Range which shows high dispersion.
- **Quartile Deviation:** Quartile deviation is the Arithmetic mean of the difference between Third Quartile and the First Quartile of the data.
- **Mean Deviation:** Mean Deviation is the value obtained by taking arithmetic mean of the deviations obtained by deducting average of data whether Mean, Median or Mode from values of data, ignoring the signs of the deviations.
- **Standard Deviation:** Standard Deviation is the square root of the Arithmetic mean of the squares of deviation of the item from its Arithmetic mean.
- **Variance:** It is square of Standard Deviation.
- **Absolute Measure:** Absolute measure of dispersion is one which is expressed in the same statistical unit in which the original values of that data are expressed. For example, if original data is represented in kilograms, the dispersion will also be represented in kilogram.

- **Relative Measure:** The relative measure of dispersion is independent of unit of measurement and is expressed in pure number. Normally it is a ratio of the dispersion to the average of the data.
- **Coefficient of Standard Deviation:** Coefficient of Deviation is the relative measure of the standard deviation and can be calculated by dividing the Value of Standard Deviation with the Arithmetic Mean. The value of coefficient always lies between 0 and 1, where 0 indicates no Standard Deviation and 1 indicated 100% standard deviation.

6.12 QUESTIONS FOR PRACTICE

Short Answer Type

Q1.What is Dispersion?

Q2.What are absolute and relative measure of dispersion?

Q3.What is range?

Q4.What is mean deviation?

Q5.What are Quartile deviations? Give its merits and limitations.

Long Answer Type

Q6.What are uses of measuring Dispersion?

Q7.What are the features of good measure of Dispersion.

Q8. Give its merits and limitations of Range.

Q9.How to calculate mean deviation. Give its merits and limitations.

Q10. What is standard deviation? How it is calculated. Give its merits and limitations.

Q11. How combined standard deviation can be calculated.

Q12. Give properties of standard deviation.

6.13 FURTHER READINGS

- J. K. Sharma, Business Statistics, Pearson Education.
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BACHELOR OF ARTS

SEMESTER – VI

QUANTITATIVE METHODS

UNIT 7: CORRELATION ANALYSIS

STRUCTURE

7.0 OBJECTIVES

After studying the Unit, students will be able to:

- Define Correlation
- Distinguish between different types of correlation
- Understand the benefits of correlation
- Find correlation using the graphic method
- Calculate correlation by Karl Pearson Method
- Measure correlation using Rank correlation method

7.1 INTRODUCTION

When we study measurement of central tendency, dispersion analysis, skewness analysis etc., we study the nature and features of data in which only one variable is involved. However, In our daily life, we come across many things in which two or more variables are involved and such variables may be related to each other. As these variables are related to each other, it is important to understand the nature of such a relation and its extent. Identification of such relations helps us in solving many problems of daily life. This is not only helpful in our daily lives but also helpful in solving many business problems.

7.2 MEANING OF CORRELATION

Correlation is a statistical technique that studies the relationship between two or more variables. It studies how two variables are related to each other. It studies how the change in the value of one variable affects the other variable, for example in our daily life we will find the relation between income and expenditure, income and demand, Price and Demand age of husband-and-wife etcetera correlation helps in understanding such relations of different variables two variables are said to be related to each other when a change in the value of one variable so results in to change in the value of other variables.

Therefore, when X and Y are related to each other, then it has four possibilities:

- X may be causing Y
- Y may be causing X
- X and Y both are bidirectionally related, i.e., X is causing Y and Y is causing X
- X and Y are related to each other through some third variable

However, correlation has nothing to do with causation. It simply attempts to find the degree of mutual association between them. Two variables might be found highly correlated, but they are not causing the change in each other. There may be a correlation due to pure chance. For example, we may find a high degree of correlation between the number of trees in a city and number of drug addicts. However, there is no theoretical base that relates these variables together Such correlation is known as Spurious Correlation or Non-sense Correlation.

According to W.I. King, "Correlation means that between two series or groups of data, there exists some causal connection."

7.3 USES OF CORRELATIONS

1. It helps us in understanding the extent and direction of the relation between two variables. It shows, whether two variables are positively correlated or negatively correlated. It also shows whether a relation between two variables is high or low.
2. Correlation also helps in the prediction of the future, for example, if we know the relation between monsoon and agricultural produce, we can predict that what will be the level of produce on basis of monsoon prediction. We can also predict the price of Agricultural Products depending on the level of produce.
3. With the help of correlation, we can find the value of one variable when the value of another variable is known. This can be done by using the statistical technique called regression analysis.
4. Correlation also helps in business and Commerce. A businessman can fix price of its product using correlation analysis. Correlation also helps him in deciding business policy.
5. Correlation also helps government in deciding its economic policy. With the help of correlation, government can study relation between various economic variables, thus government can decide their economic policies accordingly.
6. Correlation is also helpful in various statistical Analyses. Many Statistical techniques use

correlation for further analysis.

7.4 TYPES OF CORRELATION

A. Positive, Negative and No Correlation

a. Positive correlation: It is a situation in which two variables move in the same direction. In this case, if the value of one variable increases the value of the other variable also increases. Similarly, if the value of one variable decreases, the value of other variables also decreases. So, when both the variables either increase or decrease, it is known as a positive correlation. For example, we can find a Positive correlation between Income and Expenditure, Population and Demand for food products, Incomes and Savings, etc. The following data shows a positive correlation between two variables:

Height of Persons: X	158	161	164	166	169	172	174
Weight of Person: Y	61	63	64	66	67	69	72

b. Negative or Inverse Correlation: When two variables move in opposite directions from each other, it is known as negative or inverse correlation. In other words, we can say that when the value of one variable increases value of another variable decreases, it is called a negative correlation. In our life we find a negative correlation between some variables, for example, there is a negative correlation between Price and Demand, the Number of Workers and Time required to complete the work, etc. The following data shows the negative correlation between two variables:

Price of Product: X	1	2	3	4	5
The demand of Product: Y	50	45	40	35	30

c. Zero or No Correlation: When two variables do not show any relation, it is known as zero or no correlation. In other words, we can say that in the case of zero correlation, the change in the value of one variable does not affect the value of other variables. In this case, two variables are independent from each other. For example, there is zero correlation between the height of the student and the marks obtained by the student.

B. Simple and Multiple Correlation

a. Simple Correlation: When we study the relation between two variables only, it is known as simple correlation. For example, relation between income and expenditure, Price and Demand, are situations of simple correlation.

b. Multiple Correlation: Multiple correlation is a situation in which more than two variables are involved. Here relation between more than two variables is studied together, for example, if we are studying the relation between the income of the consumer, price of the product and demand for the product, it is a situation of multiple correlations.

C. Total and Partial Correlation

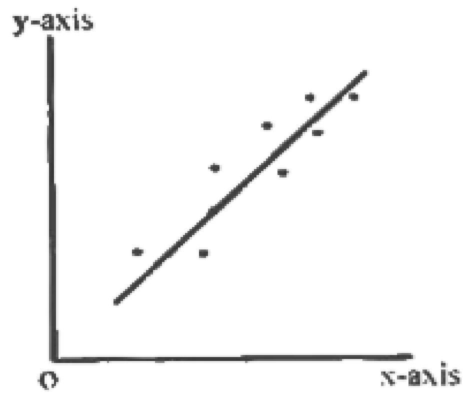
a. Total Correlation: In case we study relation of more than two variables and all the variables are taken together, it is a situation of total correlation. For example, if we are studying the relationship between the income of the consumer, the price of the product and the demand of the product, taking all the factors together it is called total correlation.

b. Partial Correlation: In the case of partial correlation more than two variables are involved, but while studying the correlation we consider only two factors assuming that the value of other factors is constant. For example, while studying the relationship between the income of the consumer, price of the product and demand for the product, we take into consideration only relation between price of the product and demand for the product assuming that the income of the consumer is constant.

D. Linear and Non-Linear Correlation

a. Linear Correlation: When the change in value of one variable results in a constant ratio of change in the value of other variables, it is called linear correlation. In such a case, if we draw the values of two variables on the graph paper, all the points on the graph paper will fall on a straight line. For example, every change in income of a consumer by Rs. 1000 results in an increase in consumption by 10 kg., which is known as linear correlation. The following data shows an example of linear correlation:

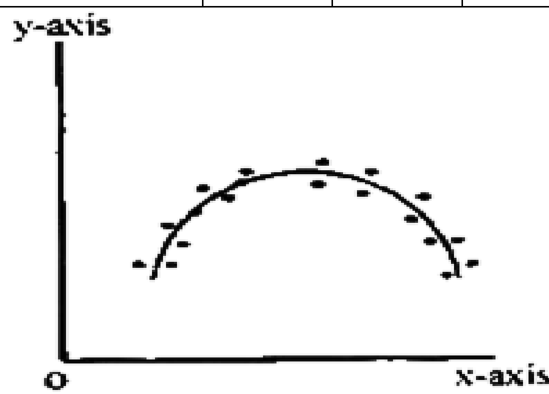
Price of Product: X	1	2	3	4	5
Demand for Product: Y	50	45	40	35	30



Linear Correlation

b. Non-Linear Correlation: When the change in value of one variable does not result in a constant ratio of change in the value of other variables, it is called nonlinear correlation. In such case, if we draw the value of two variables on the graph paper all the points will not fall in the straight line on the graph. The following data shows a nonlinear correlation between two variables:

Price of Product: X	1	2	3	4	5
Demand of Product: Y	50	40	35	32	30



Non Linear Correlation

7.4 DEGREES OF CORRELATION

Here degrees of correlation shown in the following table:

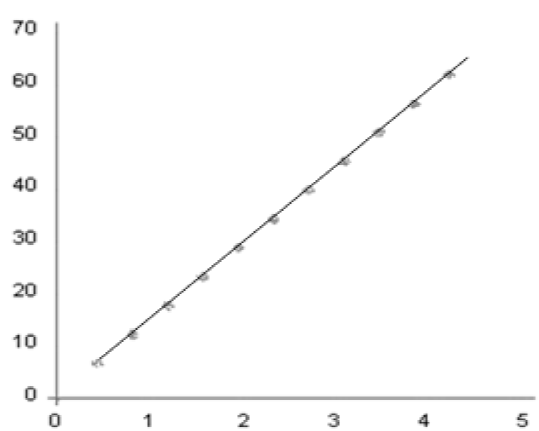
Degrees of Correlation	Positive	Negative
1. Perfect Degree	+1	-1
2. Very High Degree	+0.9 0and more	-0.9 0and more
3. High Degree	+0.75 to .90	-0.75 to .90
4. Moderate Degree	+0.50 to 0.75	-0.50 to 0.75
5. Low degree	+0.25 to 0.50	-0.25 to 0.50

6. Very Low Degree	+Less than 0.25	-Less than 0.25
7. Zero Degree	0	0

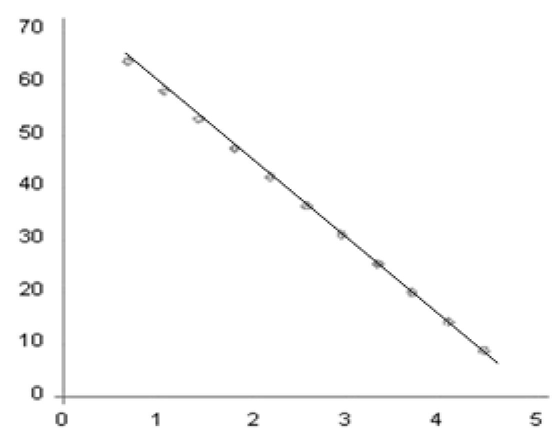
7.5 SCATTER DIAGRAM METHOD

Scatter Diagram is one of the oldest and simplest methods of measuring the correlation. This is a graphic method of measuring the correlation. This method uses a diagram representation of bivariate data to find out the degree and direction of correlation. Under this method, values of the data are plotted on a graph paper by taking one variable on the x-axis and other variables on the y-axis. Normally independent variable is shown on the x-axis whereas the value of the dependent variable is taken on the y-axis. Once all the values are drawn on the graph paper, we can find out degree of correlation between two variables by looking at direction of dots on the graph. Scatter Diagram shows whether two variables are co-related to each other or not. It also shows the direction of correlation whether positive or negative and the shows extent of correlation whether high or low. The following situations are possible in the scatter diagram.

1. **Perfect Positive Correlation**: After we plot two variables on the graph, if the points of the graph fall in a straight line that moves from the lower left-hand side to the upper corner on the right-hand side, then it is assumed that there is perfect positive correlation between the variables.
2. **Perfect Negative Correlation:** After drawing the variables on the graph, if all the points fall in a straight line but direction of the points is downward from right-hand corner to left-hand side corner, then it is assumed that there is perfect negative correlation between the variables.

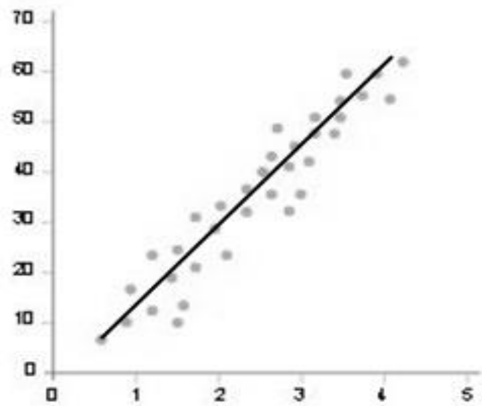


Perfect Positive Correlation

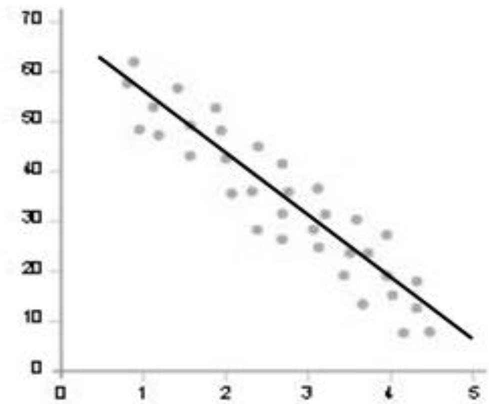


Perfect Negative Correlation

3. **High Degree of Positive Correlation**: If we draw two variables on the graph and we find that the points move in an upward direction from left-hand corner to the right-hand corner but not in a straight line, rather these are in narrow band, we can assume that there is a high degree of positive correlation between the variables.
4. **High Degree of Negative Correlation**: After plotting the dots on a graph, if we find that all the dots move downward from left-hand corner to the right-hand side corner but not in a straight line but rather in a narrow band, we can say that there is high degree of negative correlation between the variables.

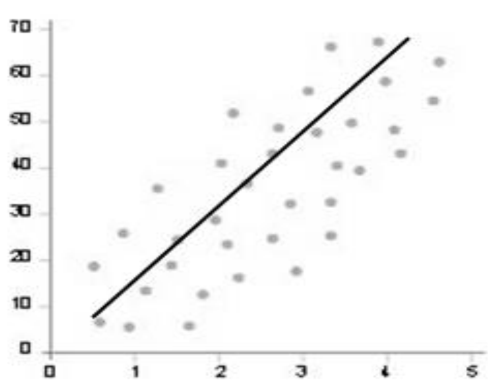


High Positive Correlation

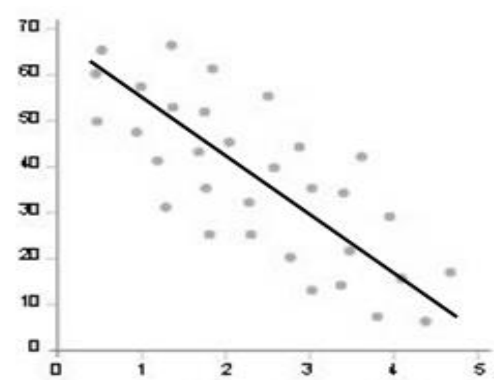


High Negative Correlation

5. **Low Degree of Positive Correlation**: In case the dots drawn on a graph paper move upward from left side to right side but the dots are widely scattered, it can be said that there is a low degree of positive correlation between the variables.
6. **Low Degree of Negative Correlation**: In case the points drawn on a graph are in a downward direction from left side to right side but the points are widely scattered, it is the situation of low degree of negative correlation between the variables.

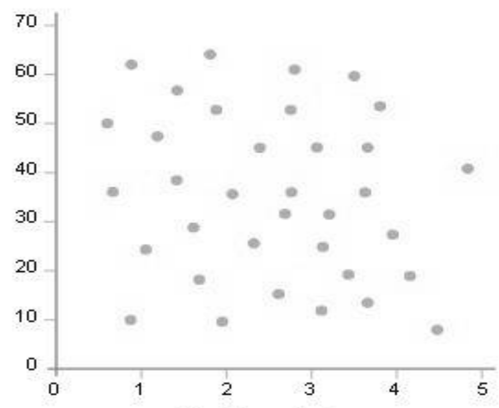


Low Positive Correlation



Low Negative Correlation

7. **Zero or No Correlation**: Sometimes find that the dots drawn on a graph paper do not move in any direction and are widely scattered in the graph paper, we can assume that there is no correlation between the two variables.



No Correlation

NOTE:

- Correlation coefficient shows the linear relationship between X and Y. Thus, even if there is a strong non-linear relationship between X and Y, the correlation coefficient may be low.
- Correlation coefficient is independent of scale and origin. If we subtract some constant from one (or both) of the variables, the correlation coefficient will remain unchanged. Similarly, if we divide one (or both) of the variables by some constant, the correlation coefficient will not change.
- Correlation coefficient varies between -1 and +1. This means r cannot be smaller than -1 and cannot be greater than +1.

The existence of a linear relationship between two variables is not to be interpreted to mean a cause-effect relationship between the two.

7.7 PROPERTIES OF CORRELATION

1. **Range:** The coefficient of Correlation always lies between -1 to +1.
2. **Degree Of Measurement**: Correlation Coefficient is independent of units of measurement.
3. **Direction**: The sign of Correlation is positive (+ve) if the values of variables move in the same direction, if -ve then the opposite direction.
4. **Symmetry**: Correlation Coefficient deals with the property of symmetry. It means $r_{xy}=r_{yx}$,

5. **Geometric Mean:** The coefficient of Correlation is also the geometric mean of two regression coefficients Rxy= bxy. byx
6. If x and y are independent then $r_{xy}=0$
7. **Change of Origin**: The correlation coefficient is independent of change of origin
8. **Change of Scale**: The correlation coefficient is independent of change in Scale
9. **Coefficient of determination**: The square of the correlation coefficient (r_{xy}) is known as the coefficient of determination.

7.8 KARL PEARSONS'S COEFFICIENT OF CORRELATION

Karl Peason's Coefficient of Correlation is the most important method of measuring the correlation. Karl Peason's Coefficient of correlation is also denoted as 'Product Moment Correlation'. The coefficient of correlation given by Karl Pearson is denoted as a symbol 'r'. It is the relative measure of finding the correlation.

According to Karl Pearson we can determine correlation by dividing the product of deviations taken from the mean of the data.

7.9 DIFFERENT METHODS TO CALCULATE COEFFICIENT OF CORRELATION

7.9.1 Direct method of calculating Correlation

Correlation can be calculated using the direct method without taking any mean. The following are the steps:

1. Take two series X and Y.
2. Find the sum of these two series denoted as $\sum X$ and $\sum Y$.
3. Take the square of all the values of the series X and series Y.
4. Find the sum of the square so calculated denoted by $\sum X^2$ and $\sum Y^2$.
5. Multiply the corresponding values of series X and Y and find the product.
6. Sum up the product so calculated denoted by $\sum XY$.
7. Apply the following formula for calculating the correlation.

Coefficient of Correlation, $\mathbf{r} = \frac{N\sum XY - (\sum X)(\sum Y)}{\sqrt{N\sum X^2 - (\sum X)^2}\sqrt{N\sum Y^2 - (\sum Y)^2}}$

Example 1: Find the coefficient of correlation

X	Y
2	4
3	5

1	**3**
5	**4**
6	**6**
4	**2**

Solution:

X	Y	X^2	Y^2	XY
2	4	4	16	8
3	5	9	25	15
1	3	1	9	3
5	4	25	16	20
6	6	36	36	36
4	2	16	4	8
$\sum X = 21$	$\sum Y = 24$	$\sum X^2 = 91$	$\sum Y^2 = 106$	$\sum XY = 90$

$$N = 6$$

Coefficient of Correlation, $r = \frac{N\sum XY - (\sum X)(\sum Y)}{\sqrt{N\sum X^2 - (\sum X)^2}\sqrt{N\sum Y^2 - (\sum Y)^2}}$

$$= \frac{6\times 90 - 21\times 24}{\sqrt{6\times 91 - (21)^2}\sqrt{6\times 106 - (24)^2}}$$

$$= \frac{540 - 504}{\sqrt{546 - 441}\sqrt{636 - 576}}$$

$$= \frac{36}{\sqrt{105}\sqrt{60}} = \frac{36}{10.246\times 7.7459}$$

$$= \frac{36}{79.31} = 0.4539$$

$\Rightarrow$ $r = 0.4539$

7.9.2 Actual Mean method of calculating Correlation

Under this Correlation is calculated by taking the deviations from actual mean of the data. The following are the steps:

1. Take two series X and Y.
2. Find the mean of both the series X and Y, denoted by $\bar{\mathbf{X}}$ and $\bar{\mathbf{Y}}$.
3. Take deviations of series X from it mean and it is denoted by 'x'.
4. Take deviations of series Y from it mean and it is denoted by 'y'.
5. Take square of deviation of series X denoted by x^2.
6. Sum up square of deviations of series X denoted by $\sum x^2$.
7. Take square of deviation of series Y denoted by y^2.
8. Sum up square of deviations of series Y denoted by $\sum y^2$.

9. Find the product of x and y and it is denoted by xy.

10. Find the sum of 'xy 'it is denoted by $\sum$ xy

11. Apply the following formula for calculating the correlation.

$$\mathbf{r} = \frac{\sum \mathbf{xy}}{\sqrt{\sum \mathbf{x^2}}\sqrt{\sum \mathbf{y^2}}} = \frac{\sum(\mathbf{X}-\overline{\mathbf{X}})(\mathbf{Y}-\overline{\mathbf{Y}})}{\sqrt{\sum(\mathbf{X}-\overline{\mathbf{X}})^2}\sqrt{(\mathbf{Y}-\overline{\mathbf{Y}})^2}}$$

Example 2. Calculate Karl Pearson's coefficient of correlation

X	**50**	**50**	**55**	**60**	**65**	**65**	**65**	**60**	**60**	**50**
Y	**11**	**13**	**14**	**16**	**16**	**15**	**15**	**14**	**13**	**13**

Solution: When deviations are taken from actual arithmetic mean, 'r' is given by

$$r = \frac{\sum xy}{\sqrt{\sum x^2}\sqrt{\sum y^2}} = \frac{\sum(X-\overline{X})(Y-\overline{Y})}{\sqrt{\sum(X-\overline{X})^2}\sqrt{(Y-\overline{Y})^2}}$$

Where $x = X - \overline{X} =$ Deviation from A. M. of X series

$y = Y - \overline{Y} =$ Deviation from A. M. of Y series

X	Y	$x = (X - \overline{X})$	x^2	$y = (Y - \overline{Y})$	y^2	xy
50	11	−8	64	−3	9	24
50	13	−8	64	−1	1	8
55	14	−3	9	0	0	0
60	16	2	4	2	4	4
65	16	7	49	2	4	14
65	15	7	49	1	1	7
65	15	7	49	1	1	7
60	14	2	4	0	0	0
60	13	2	4	−1	1	−2
50	13	−8	64	−1	1	8
$\sum X = 580$	$\sum Y = 140$		$\sum x^2 = 360$		$\sum y^2 = 22$	$\sum xy = 70$

Here, $N = 10$

A. M. of X series, $\overline{X} = \frac{\sum X}{N} = \frac{580}{10} = 58$

A. M. of Y series, $\overline{Y} = \frac{\sum Y}{N} = \frac{140}{10} = 14$

Coefficient of Correlation, $r = \frac{\sum xy}{\sqrt{\sum x^2}\sqrt{\sum y^2}} = \frac{70}{\sqrt{360 \times 22}} = \frac{70}{\sqrt{7920}} = 0.7866$

$\Rightarrow \quad r = 0.7866$

7.9.3 Assumed Mean method of calculating Correlation

Under this Correlation is calculated by taking the deviations from assumed mean of the data. Following are the steps:

1. Take two series X and Y.
2. Take any value as assumed mean for series X.
3. Take deviations of series X from its assumed mean and it is denoted by 'dx'.
4. Find sum of deviations denoted by $\sum dx$.
5. Take square of deviation of series X denoted by dx^2
6. Sum up square of deviations of series X denoted by $\sum dx^2$.
7. Take any value as assumed mean for series Y.
8. Take deviations of series Y from its assumed mean and it is denoted by 'dy'.
9. Find sum of deviations of series Y denoted by $\sum dy$.
10. Take square of deviation of series Y denoted by dy^2
11. Sum up square of deviations of series Y denoted by $\sum dy^2$.
12. Find the product of dx and dy and it is denoted by dxdy.
13. Find the sum of 'dxdy' it is denoted by $\sum dxdy$
14. Apply the following formula for calculating the correlation.

$$\mathbf{r} = \frac{\mathbf{N \sum dxdy - (\sum dx)(\sum dy)}}{\sqrt{\mathbf{N \sum dx^2 - (\sum dx)^2}}\sqrt{\mathbf{N \sum dy^2 - (\sum dy)^2}}}$$

Example 3. Compute coefficient of correlation from the following figures

City	**A**	**B**	**C**	**D**	**E**	**F**	**G**
Population (in '000)	**78**	**25**	**16**	**14**	**38**	**61**	**30**
Accident Rate (Per million)	**80**	**62**	**53**	**60**	**62**	**69**	**67**

Solution: Here, $N = 7$

Coefficient of Correlation, r is given by

$$r = \frac{N \sum dxdy - (\sum dx)(\sum dy)}{\sqrt{N \sum dx^2 - (\sum dx)^2}\sqrt{N \sum dy^2 - (\sum dy)^2}}$$

Where dx = Deviations of terms of X series from assumed mean $A_X = X - A_X$

dy = Deviations of terms of Y series from assumed mean $A_Y = Y - A_Y$

X	Y	$dx = X - A_X$ $A_X = 38$	$dy = Y - A_Y$ $A_Y = 67$	dx^2	dy^2	dxdy
70	80	32	13	1024	169	416

25	62	-13	-5	169	25	65
16	53	-22	-14	482	196	308
14	60	-24	-7	576	49	168
38	62	0	-5	0	25	0
61	69	23	2	529	4	46
30	67	-8	0	64	0	0
		$\sum dx = -12$	$\sum dy = -16$	$\sum dx^2 = 2846$	$\sum dy^2 = 468$	$\sum dxdy = 1003$

Here, $N = 7$

$\therefore$ Coefficient of Correlation, $r = \frac{7\times1003-(-12)(-16)}{\sqrt{7\times2846-(-12)^2}\sqrt{7\times468-(-16)^2}}$

$$= \frac{7021-192}{\sqrt{19{,}922-144}\sqrt{3276-256}}$$

$$= \frac{6829}{\sqrt{19{,}778}\sqrt{3020}} = 0.8837$$

$$r = 0.8837$$

7.9.4 Step Deviation method of calculating Correlation

Under this method assumed mean is taken but the difference is that after taking the deviation, these are divided by some common factor to get the step deviations. The following are the steps:

1. Take two series X and Y.
2. Take any value as assumed mean for series X.
3. Take deviations of series X from its assumed mean and it is denoted by ‘dx’.
4. Divide the value of ‘dx’ so obtained by some common factor to get $\mathbf{dx'}$
5. Find sum of deviations denoted by $\sum\mathbf{dx'}$.
6. Take square of deviation of series X denoted by $\mathbf{dx'}^2$
7. Sum up square of deviations of series X denoted by $\sum \mathbf{dx'}^2$.
8. Take any value as assumed mean for series Y.
9. Take deviations of series Y from its assumed mean and it is denoted by ‘dy’.
10. Divide the value of ‘dy’ so obtained by some common factor to get $\mathbf{dy'}$
11. Find sum of deviations of series Y denoted by $\sum\mathbf{dy'}$.
12. Take square of deviation of series Y denoted by $\mathbf{dy'}^2$
13. Sum up square of deviations of series Y denoted by $\sum \mathbf{dy'}^2$.
14. Find the product $\mathbf{dx'}$ of and $\mathbf{dy'}$ and it is denoted by $\mathbf{dx'}\ \mathbf{dy'}$.

15. Find the sum of 'dxdy' it is denoted by $\sum \mathbf{dx'\ dy'}$

16. Apply the following formula for calculating the correlation.

$$\textbf{Coefficient of Correlation, r} = \frac{N\sum dx'dy' - (\sum dx')(\sum dy')}{\sqrt{N\sum dx'^2 - (\sum dx')^2}\sqrt{N\sum dy'^2 - (\sum dy')^2}}$$

Example 4. Find the coefficient of correlation by Karl Pearson's method

Price (Rs.)	5	10	15	20	25
Demand (kg)	**40**	**35**	**30**	**25**	**20**

Solution:

X	Y	$dx = X - A$ $A = 15$	$dx' = \frac{dx}{C_1}$ $C_1 = 5$	$dy = Y - B$ $B = 30$	$dy' = \frac{dy}{C_1}$ $C_2 = 5$	dx'^2	dy'^2	$dx'dy'$
5	40	−10	−2	10	2	4	4	−4
10	35	−5	−1	5	1	1	1	−1
15	30	0	0	0	0	0	0	0
20	25	5	1	−5	−1	1	1	−1
25	20	10	2	−10	−2	4	4	−4
			$\sum dx' = 0$		$\sum dy' = 0$	$\sum dx'^2 = 10$	$\sum dy'^2 = 10$	$\sum dx'dy' = -10$

Here, $N = 5$

$$\text{Coefficient of Correlation, r} = \frac{N\sum dx'dy' - (\sum dx')(\sum dy')}{\sqrt{N\sum dx'^2 - (\sum dx')^2}\sqrt{N\sum dy'^2 - (\sum dy')^2}}$$

$$= \frac{5\times(-10) - 0\times 0}{\sqrt{5\times 10 - 0^2}\sqrt{5\times 10 - 0^2}}$$

$$= \frac{-50}{\sqrt{50}\times\sqrt{50}} = -1$$

$\Rightarrow \qquad r = -1$

7.9.5 Calculating Correlation with help of Standard Deviations

Under this method assumed mean is taken but the difference is that after taking the deviation, these are divided by some common factor to get the step deviations. Following are the steps:

1. Take two series X and Y.
2. Find the mean of both the series X and Y, denoted by $\mathbf{\bar{X}}$ and $\mathbf{\bar{Y}}$.
3. Take deviations of series X from it mean and it is denoted by 'x'.
4. Take deviations of series Y from it mean and it is denoted by 'y'.
5. Find the product of x and y and it is denoted by xy.
6. Find the sum of 'xy 'it is denoted by $\sum xy$

7. Calculate the standard deviation of both series X and Y.
8. Apply the following formula for calculating the correlation.

$$\mathbf{r = \frac{\sum xy}{N\sigma_X\sigma_Y}}$$

Example 5. Given

No. of pairs of observations = 10

$\sum xy = 625$

X Series Standard Deviation = 9

Y Series Standard Deviation =8

Find 'r'.

Solution: We are given that

$N = 10$, $\sigma_X = 9$ $\sigma_Y = 8$ and $\sum xy = 625$

Now $r = \frac{\sum xy}{N\sigma_X\sigma_Y}$

$= \frac{625}{10\times9\times8} = \frac{625}{720} = 0.868$

$\Rightarrow$ $r = +.868$

Example 6. Given

No. of pairs of observations = 10

X Series Arithmetic Mean = 75

Y Series Arithmetic Mean = 125

X Series Assumed Mean = 69

Y Series Assumed Mean = 110

X Series Standard Deviation = 13.07

Y Series Standard Deviation = 15.85

Summation of products of corresponding deviation of X and Y series = 2176

Find 'r'.

Solution: We are given that

$N = 10$, $\overline{X} = 75$, $A_X = 69$, $\sigma_X = 13.07$

$\overline{Y} = 125$, $A_Y = 110$, $\sigma_Y = 15.85$

and $\sum xy = 2176$

Now $r = \frac{\sum xy - N(\overline{X}-A_X)(\overline{Y}-A_Y)}{N\sigma_X\sigma_Y}$

$= \frac{2176-10(75-69)(125-110)}{10\times13.07\times15.85} = \frac{2176-900}{2071.595}$

$= 0.6159 \approx 0.616$

$\Rightarrow \qquad r = +0.616$

Example 7. A computer while calculating the coefficient of correlation between the variables X and Y obtained the values as

$$\mathbf{N = 6, \quad \sum X = 50, \quad \sum X^2 = 448}$$
$$\mathbf{\sum Y = 106, \quad \sum Y^2 = 1896, \quad \sum XY = 879}$$

But later on, it was found that the computer had copied down two pairs of observations as

X	Y
5	15
10	18

While the correct values were

X	Y
6	18
10	19

Find the correct value of correlation coefficient.

Solution**:** Incorrect value of $\sum X = 50$

$\therefore$ Correct value of $\sum X = 50 - 5 - 10 + 6 + 10$
$= 51$

Incorrect value of $\sum Y = 106$

$\therefore$ Correct value of $\sum Y = 106 - 15 - 18 + 18 + 19$
$= 110$

Incorrect value of $\sum X^2 = 448$

$\therefore$ Correct value of $\sum X^2 = 448 - 5^2 - (10)^2 + 6^2 + (10)^2$
$= 459$

Incorrect value of $\sum Y^2 = 1896$

$\therefore$ Correct value of $\sum Y^2 = 1896 - 15^2 - (18)^2 + (18)^2 + 19^2$
$= 2032$

Incorrect value of $\sum XY = 879$

$\therefore$ Correct value of $\sum XY = 879 - (5 \times 15) - (10 \times 18) + (6 \times 18) + (10 \times 19)$
$= 952$

Thus, the corrected value of coefficient of correlation

$$= \frac{N\sum XY - (\sum X)(\sum Y)}{\sqrt{N\sum X^2 - (\sum X)^2}\sqrt{N\sum Y^2 - (\sum Y)^2}}$$

$$= \frac{6 \times 952 - 51 \times 110}{\sqrt{6 \times 459 - (51)^2}\sqrt{6 \times 2032 - (110)^2}}$$

$$= \frac{5712 - 5610}{\sqrt{2754 - 2601}\sqrt{12{,}192 - 12{,}100}}$$

$$= \frac{102}{\sqrt{153}\sqrt{92}} \qquad = \frac{102}{12.369 \times 9.59}$$

$$= \frac{102}{118.618} = 0.8599$$

$\Rightarrow \qquad r = +0.8599$

7.9.6 Limitations of Karl Pearson's Coefficient of Correlation

1. It is comparatively difficult to calculate.
2. It is time consuming method.
3. It is based on unrealistic assumptions.
4. It is affected by extreme values.
5. It cannot be applied on qualitative data.

TEST YOUR UNDERSTANDING (A)

1. From the following data of prices of product X and Y draw scatter diagram.

	1st	2nd	3rd	4th	5th	6th	7th	8th	9th	10th
Price of X	60	65	65	70	75	75	80	85	80	100
Price of Y	120	125	120	110	105	100	100	90	80	60

2. Calculate Karl Pearson's coefficient of correlation

X	21	22	23	24	25	26	27	28	29	30
Y	46	42	38	34	30	26	22	18	14	10

3. Calculate Karl Pearson's coefficient between X and Y

X	42	44	58	55	89	98	66
Y	56	49	53	58	65	76	58

4. Find correlation between marks of subject A Subject B

Subject A	24	26	32	33	35	30
Subject B	15	20	22	24	27	24

5. Find correlation between Height of Mother and Daughter

Height of Mother (Inches)	54	56	56	58	62	64	64	66	70	70
Height of Daughter (Inches)	46	50	52	50	52	54	56	58	60	62

6. What is the Karl Pearson's coefficient of correlation if $\sum xy = 40$, $n = 100$, $\sum x^2 = 80$ and $\sum y^2 = 20$.

7. Calculate the number of items for which r = 0.8, $\sum xy = 200$. Standard deviation of y = 5 and $\sum x^2 = 100$ where x and y denote the deviations of items from actual means.

8. Following values were obtained during calculation of correlation:

N = 30; $\sum X = 120$ $\sum X^2 = 600$ $\sum Y = 90$ $\sum Y^2 = 250$ $\sum XY = 335$

Later found that two pairs were taken wrong which are as follows:

pairs of observations as:	(X, Y):	(8, 10)	(12, 7)
While the correct values were:	(X, Y):	(8, 12)	(10, 8)

Find correct correlation.

9. From the data given below calculate coefficient of correlation.

	X series	Y series
Number of items	8	8
Mean	68	69
Sum of squares of deviation from mean	36	44
Sum, of product of deviations x and y from means	24	24

10. Find the correlation between age and playing habits from the following data :

Age	15	16	17	18	19	20
No of students	20	270	340	360	400	300
Regular players	150	162	170	180	180	120

Answers

2) -1	4) .92	6) 1,	8) -.4311	10) -.94
3) .9042	5) .95	7) 25	9) .603	

7.10 SPEARMAN'S RANK CORRELATION

Karl Peason's Coefficient of Correlation is very useful if data is quantitative, but in case of qualitative data it is a failure. Spearman's Rank correlation is a method that can calculate correlation both from quantitative and qualitative data if the data is ranked like in singing contest we rank the participants as one number, two number or three number etc. This method was given by Charles Edward Spearman in 1904. In this method we give Rank to the data and with help of such ranks, correlation is calculated.

7.10.1 Features of Spearman's Rank correlation

1. The coefficient of correlation may be positive or negative.
2. The value of coefficient of correlation always lies between -1 and + 1. -1 refers to 100% negative correlation, plus one refers to 100% positive correlation, and zero refers to no correlation between the items.
3. This method is based ranks of the data.
4. Sum of difference between ranks in this method is always zero i.e., $\sum D = 0$.
5. There is no assumption of normal distribution in this method.
6. In case all the ranks of the two series are same the value of $\sum D^2 = 0$, it shows that there is perfect positive correlation between the data.

7.10.2 Spearman's Rank Correlation when ranks are given

1. Calculate the difference between ranks of both the series denoted by $\sum D$.
2. Take square of deviations and calculate the value of D^2.
3. Calculate sum of square of deviations denoted by $\sum D^2$.
4. Apply following formula.

Example 9. Following are given the ranks of 8 pairs. Find 'r'

Rank X	6	4	8	2	7	5	3	1
Rank Y	4	8	7	3	6	5	1	2

Solution:

Rank X	Rank Y	Difference of Ranks D	D^2
6	4	+2	4
4	8	−4	16
8	7	−1	1
2	3	−1	1
7	6	+1	1
5	5	0	0
3	1	+2	4
1	2	−1	1
N = 8			$\sum D^2 = 28$

Coefficient of Rank Correlation, $r = 1 - \frac{6\sum D^2}{N(N^2-1)}$

$$= 1 - \frac{6 \times 28}{8(8^2-1)}$$

$$= 1 - \frac{168}{8(64-1)} = 1 - \frac{168}{8(63)}$$

$$= 1 - \frac{168}{504} = 1 - 0.33 = 0.67$$

⇒ Rank Correlation Coefficient = 0.67

Example 10. In a beauty contest, three judges gave the following ranks to 10 contestants. Find out which pair of judges agree or disagree the most.

Judge 1	**5**	**1**	**6**	**3**	**8**	**7**	**10**	**9**	**2**	**4**
Judge 2	**9**	**7**	**10**	**5**	**8**	**4**	**3**	**6**	**1**	**2**
Judge 3	**6**	**4**	**7**	**10**	**5**	**3**	**1**	**9**	**2**	**8**

Solution:

Ranks by	$D_1 =$	${D_1}^2$	$D_2 =$	${D_2}^2$	$D_3 =$	${D_3}^2$

Judge 1 R_1	Judge 2 R_2	Judge 3 R_3	$R_1 - R_2$		$R_2 - R_3$		$R_1 - R_3$	
5	9	6	−4	16	3	9	−1	1
1	7	4	−6	36	3	9	−3	9
6	10	7	−4	16	3	9	−1	1
3	5	10	−2	4	−5	25	−7	49
8	8	5	0	0	3	9	3	9
7	4	3	+3	9	1	1	4	16
10	3	1	+7	49	2	4	9	81
9	6	9	+3	9	−3	9	0	0
2	1	2	+1	1	−1	1	0	0
4	2	8	+2	4	−6	36	−4	16
				$\sum D_1^2 = 144$		$\sum D_2^2 = 112$		$\sum D_3^2 = 182$

Now $r_{12} = 1 - \frac{6\sum D_1^2}{N(N^2-1)}$

$= 1 - \frac{6\times144}{10(10^2-1)}$

$= 1 - \frac{864}{10(100-1)} \quad = 1 - \frac{864}{10(99)}$

$= 1 - \frac{864}{990} = 1 - 0.873 = 0.127$

∴ $r_{12} = +0.127 \Rightarrow$ Low degree +ve correlation

$r_{23} = 1 - \frac{6\sum D_2^2}{N(N^2-1)}$

$= 1 - \frac{6\times112}{10(10^2-1)} \quad = 1 - \frac{672}{10(100-1)}$

$= 1 - \frac{672}{10(99)} \quad = 1 - \frac{672}{990}$

$= 1 - 0.679 = 0.321$

∴ $r_{23} = +0.321 \Rightarrow$ Moderate degree +ve correlation

Similarly, $r_{31} = 1 - \frac{6\sum D_3^2}{N(N^2-1)}$

$= 1 - \frac{6\times182}{10(10^2-1)} \quad = 1 - \frac{1092}{10(100-1)}$

$= 1 - \frac{1092}{10(99)} \quad = 1 - \frac{1092}{990}$

$= 1 - 1.103 = -0.103$

∴ $r_{31} = -0.103 \Rightarrow$ Low degree −ve correlation

⇒ Since r_{23} is highest, so 2nd and 3rd judges agree the most.

Also, r_{31} being lowest, 3rd and 1st judges disagree the most.

7.10.3 Spearman's Rank Correlation when ranks are not given

1. Assign the ranks in descending order to series X by giving first rank to highest value and second rank to value lower than higher value and so on.
2. Similarly assign the ranks to series Y.
3. Calculate the difference between ranks of both the series denoted by $\sum D$.
4. Take square of deviations and calculate the value of D^2.
5. Calculate sum of square of deviations denoted by $\sum D^2$.
6. Apply following formula.

Example 11. Following are the marks obtained by 8 students in Maths and Statistics. Find the Rank Correlation Coefficient.

Marks in Math's	**60**	**70**	**53**	**59**	**68**	**72**	**50**	**54**
Marks in stats	**44**	**74**	**54**	**64**	**84**	**79**	**53**	**66**

Solution:

X	Ranks R_1	Y	Ranks R_2	Difference of Ranks $D = R_1 - R_2$	D^2
60	4	44	8	-4	16
70	2	74	3	-1	1
53	7	54	6	$+1$	1
59	5	64	5	0	0
68	3	84	1	$+2$	4
72	1	79	2	-1	1
50	8	53	7	$+1$	1
54	6	66	4	$+2$	4
					$\sum D^2 = 28$

Here $N = 8$

$\Rightarrow$ Rank Coefficient of Correlation, $r = 1 - \frac{6\sum D^2}{N(N^2-1)}$

$$= 1 - \frac{6 \times 28}{8(8^2-1)}$$

$$= 1 - \frac{168}{8(64-1)}$$

$$= 1 - \frac{168}{8(63)}$$

$$= 1 - \frac{168}{504}$$

$$= 1 - 0.33 = 0.67$$

$\Rightarrow$ Rank Correlation Coefficient $= 0.67$

7.10.4 Spearman's Rank Correlation when there is repetition in ranks

1. Assign the ranks in descending order to series X by giving first rank to highest value and the second rank to value lower than higher value and so on. If two items have same value, assign the average rank to both item. For example, two equal values have ranked at 5^{th} place than rank to be given is 5.5 to both i.e., mean of 5^{th} and 6^{th} rank. ($\frac{5+6}{2}$).
2. Similarly assign the ranks to series Y.
3. Calculate the difference between ranks of both the series denoted by $\sum D$.
4. Take square of deviations and calculate the value of D^2.
5. Calculate sum of square of deviations denoted by $\sum D^2$.
6. Apply following formula.

$$\mathbf{r = 1 - \frac{6\left\{\sum D^2 + \frac{1}{12}(m_1{}^3 - m_1) + \frac{1}{12}(m_2{}^3 - m_2)\right\}}{N(N^2 - 1)}}$$

Where m = no. of times a particular item is repeated.

Example 12. Find the Spearman's Correlation Coefficient for the data given below

X	**110**	**104**	**107**	**82**	**93**	**93**	**115**	**95**	**93**	**113**
Y	**80**	**78**	**90**	**75**	**81**	**70**	**87**	**78**	**73**	**85**

Solution: Here, in X series the value 93 occurs thrice $(m_1 = 3)$, i.e. at 7^{th}, 8^{th} and 9^{th} rank. So, all the three values are given the same average rank, i.e. $\frac{7+8+9}{3} = 8^{th}$ rank.

Similarly, in Y series the value 78 occurs twice $(m_2 = 2)$, i.e. at 6^{th} and 7^{th} rank. So, both the values are given the same average rank, i.e. $\frac{6+7}{2} = 6.5^{th}$ rank.

X	Ranking of X R_1	Y	Ranking of Y R_2	Difference of Ranks $D = R_1 - R_2$	D^2
110	3	80	5	−2	4
104	5	78	6.5	−1.5	2.25
107	4	90	1	+3	9
82	10	75	8	+2	4
93	8	81	4	+4	16
93	8	70	10	−2	4
115	1	87	2	−2	1
95	6	78	6.5	−0.5	0.25
93	8	73	9	−1	1
113	2	85	3	−1	1
					$\sum D^2 = 42.5$

Here $N = 10$

Spearman's Rank Correlation Coefficient, $r = 1 - \frac{6\left\{\sum D^2 + \frac{1}{12}(m_1{}^3 - m_1) + \frac{1}{12}(m_2{}^3 - m_2)\right\}}{N(N^2 - 1)}$

i.e. $$r = 1 - \frac{6\left\{42.50+\frac{1}{12}(3^3-3)+\frac{1}{12}(2^3-2)\right\}}{10(10^2-1)}$$

$$= 1 - \frac{6\left\{42.50+\frac{24}{12}+\frac{6}{12}\right\}}{10(100-1)} = 1 - \frac{6\left\{42.50+2+\frac{1}{2}\right\}}{10\times 99}$$

$$= 1 - \frac{6\{42.5+2.5\}}{990} \quad = 1 - \frac{6\times 45}{990}$$

$$= 1 - 0.2727 = 0.7273$$

$\Rightarrow$ Rank Correlation Coefficient $= 0.7273$

Example 13. The rank correlation coefficient between the marks obtained by ten students in Mathematics and Statistics was found to be 0. 5. But later on, it was found that the difference in ranks in the two subjects obtained by one student was wrongly taken as 6 instead of 9. Find the correct rank correlation.

Solution: Given $N = 10$, Incorrect $r = 0.5$

We know that

Rank Correlation Coefficient, $r = 1 - \frac{6\sum D^2}{N(N^2-1)}$

$\Rightarrow$ $$0.5 = 1 - \frac{6\sum D^2}{10(10^2-1)} = 1 - \frac{6\sum D^2}{10\times 99}$$

$\Rightarrow$ Incorrect $\sum D^2 = \frac{990}{6} \times 0.5 = 82.5$

$\therefore$ The corrected value of $\sum D^2 = 82.5 - 6^2 + 9^2$

$$= 82.5 - 36 + 81 = 127.5$$

$\therefore$ Correct Rank Correlation Coefficient, $r = 1 - \frac{6\times 127.5}{10(10^2-1)}$

$$= 1 - \frac{765}{10(100-99)}$$

$$= 1 - \frac{765}{10\times 99}$$

$$= 1 - \frac{765}{990}$$

$$= 1 - 0.7727$$

$$= 0.2273$$

7.10.5 Limitations of Spearman's Rank Correlation

1. It cannot deal with grouped data.
2. If large data is there, it is difficult to apply this method.
3. It cannot be applied further algebraic treatment.
4. Combined correlation cannot be calculated.
5. It gives only approximate correlation; it is not based on actual values.

TEST YOUR UNDERSTANDING (B)

1. Find Rank correlation on base of following data.

X	78	36	98	25	75	82	90	62	65	39
Y	84	51	91	60	68	62	86	58	53	47

In Dance competition following ranks were given by 3 judges to participants. Determine which two judges have same preference for music:

1st Judge	1	6	5	10	3	2	4	9	7	8
2nd Judge	3	5	8	4	7	10	2	1	6	9
3rd Judge	6	4	9	8	1	2	3	10	5	7

3. Find Rank correlation on base of following data.

X	25	30	38	22	50	70	30	90
Y	50	40	60	40	30	20	40	70

4. Find Rank correlation on base of following data.

X	63	67	64	68	62	66	68	67	69	71
Y	66	68	65	69	66	65	68	69	71	70

Answers

1) .82	2) I and II -.2121, II and III -.297, I and III .6364, so judge I and III	3) 0	4) 0.81

7.11 SUM UP

- Correlation shows the relation between two or more variables.
- The value of the coefficient of correlation always lies between -1 and +1.
- Correlation may be positive or negative.
- Correlation may be linear or nonlinear.
- Karl Person's coefficient of correlation is the most popular method of correlation.
- It can deal only with quantitative data.
- Spearman's Rank correlation calculated correlation on the basis of ranks given to data.
- It can deal with qualitative data also.

7.12 KEY TERMS

- **Correlation:** Correlation is a statistical technique which studies the relation between two or more variables. It studies that how to variables are related to each other.
- **Positive correlation**: It is a situation in which two variables move in the same direction. In this case if the value of one variable increases the value of other variable also increase.

Similarly, if the value of one variable decrease, the value of other variable also decrease.

- **Negative or Inverse Correlation:** When two variables move in opposite direction from each other, it is known as negative or inverse correlation. In other words, we can say that when the value of one variable increase value of other variable decrease, it is called negative correlation.
- **Linear Correlation:** When the change in value of one variable results into constant ratio of change in the value of other variable, it is called linear correlation. In such case if we draw the values of two variables on the graph paper, all the points on the graph paper will fall on a straight line.
- **Non - Linear Correlation:** When the change in value of one variable does not result into constant ratio of change in the value of other variable, it is called non-linear correlation. In such case, if we draw the value of two variables on the graph paper all the points will not fall in the straight line on the graph.
- **Simple Correlation:** When we study relation between two variables only, it is known as simple correlation. For example, relation between income and expenditure, Price and Demand, are situations of simple correlation.
- **Multiple Correlation:** Multiple correlation is a situation in which more than two variables are involved. Here relation between more than two variables are studied together, for example if we are studying the relation between income of the consumer, price if the product and demand of the product, it is a situation of multiple correlation.

7.13 QUESTIONS FOR PRACTICE

Short Answer Type

Q1. What is Correlation?

Q2. What are the uses of measuring correlation?

Q3. What do you mean by scatter diagram?

Q4. Give Karl Persons method of calculating correlation.

Q5. What is positive and negative correlation?

Q6. What is Spearman's Rank correlation?

Long Answer Type

Q1. Explain the properties of the Correlation Coefficient.

Q2. Give different types of correlation.

Q3. What are the various degrees of correlation coefficient?

Q4. Give Karl Pearson's coefficient of correlation in the case of actual and assumed mean.

Q5. What are the limitations of Karl Pearson's method?

Q6. In the case of repeated ranks, how would you determine Spearman's Rank correlation?

Q7. What are the limitations of Spearman's Rank correlation?

7.14 SUGGESTED READINGS

- J. K. Sharma, Business Statistics, Pearson Education.
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BACHELOR OF ARTS

SEMESTER – VI

QUANTITATIVE METHODS

UNIT 8: SIMPLE REGRESSION ANALYSIS

STRUCTURE

8.0 OBJECTIVES

After studying the Unit, students will be able to

- Describe what is regression.
- Distinguish between different types of Regression.
- Understand the benefits of Regression.
- Find Regression using various methods.

- Show how correlation and regression are related.
- Understand the properties of regression coefficients.

8.1 INTRODUCTION

Statistics has many applications in our life whether it's business life or our routine life. Many techniques in statistics can help us in prediction. Regression is one such technique. In the literary meaning the term ‘Regression’ is 'going back', or 'stepping down’. So, regression analysis is a tool in statistics that can help in the prediction of one variable when the value of other variable is known if there exists any close relation between two or more variables, though such relation may be positive or negative. The technique of Regression can be widely used as a powerful tool in almost all fields whether science, social science, business, etc. However, particularly, in the fields of business and management this technique is very useful for studying the relationship between different variables such as Price and Demand, Price and Supply, Production and Consumption, Income and Consumption, Income and Savings, etc.

8.3 MEANING OF REGRESSION ANALYSIS

When we find a regression between two or more variables, we try to understand the behavior of one variable with the help movement of the other variable in a particular direction. For example, if the correlation coefficient between the value of sales and amount spent on advertisement is +0.9, it means that if advertisement expenditure is increased, Sales are also likely to increase, as there is a very high positive relation between the two variables. However, correlation only tells the relation between two variables, but it does not tell the extent to which a change in one variable will affect the change in other variables. For this purpose, we have to calculate the co-efficient of Regression. The regression Coefficient is a statistical measure that tries to find out the value of one variable known as a dependent variable when the value of another variable known as an independent variable is known. Thus, in the case of two variables, like Advertisement expenditure and amount of Sales, we can estimate the likely amount of Sales if the value of Advertisement expenditure is given. Similarly, we can predict the value of Advertisement expenditure required, to achieve a particular amount of Sales. This can be done using the two regression coefficients

Some of these definitions are given below:

1. According to Sir Francis Galton, the term regression analysis is defined as "the law of regression that tells heavily against the full hereditary transmission of any gift, the more bountifully the parent is gifted by nature, the rarer will be his good fortune if he begets a son who is richly endowed as himself, and still more so if he has a son who is endowed yet more largely."

2. In the words of Ya Lun Chou, "Regression analysis attempts to establish the nature of the relationship between variables that is to study the functional relationship between the variables and thereby provide a mechanism for prediction or forecasting".

8.4 BENEFITS / USES OF REGRESSION ANALYSIS

The benefits of Regression analysis are outlined as under:

1. **Forecasting or Prediction**: Regression provides a relationship between two or more variables that are related to each other. So, with the help of this technique, we can easily forecast the values of one variable that is unknown from the values of another variable that is known.
2. **Cause and Effect Relationship:** This analysis helps in finding the cause-and-effect relationship between two or more variables. It is a powerful tool for measuring the cause-and-effect relationship among economic variables. In the field of economics, it is very beneficial in the estimation of Demand, Production, Supply etc.
3. **Measuring Error in Estimation:** Regression helps in measuring errors in estimates made through the regression lines. In case the point of Regression line is less scattered around the relevant regression line, it means there are less chances of error but if there are more scattered around line of regression, it means there are more chances of error.
4. **Finding Correlation Coefficient between two variables:** Regression provides a measure of coefficient of correlation between the two variables. We can calculate correlation by taking the square root of the product of the two regression coefficients.
5. **Usefulness in Business and Commerce**: Regression is a very powerful tool of statistical analysis in the field of business and commerce as it can help businessmen in the prediction of various values such as demand, production etc.
6. **Useful in day-to-day life:** This technique is very useful in our daily life as it can predict various factors such as birth rate, death rate, etc.
7. **Testing Hypothesis:** The technique of regression can be used in testing the validity of economic theory or testing any hypothesis.

8.5 LIMITATIONS OF REGRESSION ANALYSIS

Though Regression is a wonderful statistical tool, still it suffers from some limitations. The following are the limitations of Regression analysis:

1. Regression analysis assumes that there exists a cause-and-effect relationship between the variables and such a relation is not changeable. This assumption may not always hold good and thus could give misleading results.
2. Regression analysis is based on some limited data available. However, as the values are based on limited data it may give misleading results.
3. Regression analysis involves very lengthy and complicated steps of calculations and analysis. A layman may not be in a position to use this technique.
4. Regression analysis can be used in the case of quantitative data only. It cannot be used where data is qualitative such as hard work, beauty etc.

8.6 DIFFERENT TYPES OF REGRESSION ANALYSIS

1. Simple and Multiple Regression

- **Simple Regression**: When there are only two variables under study it is known as a simple regression. For example, we are studying the relation between Sales and Advertising expenditure. If we consider sales as Variable X and advertising as variable Y, then the X = a+bY is known as the regression equation of X on Y where X is the dependent variable and Y and the independent variable. In other words, we can find the value of variable X (Sales) if the value of Variable Y (Advertising) is given.
- **Multiple Regression:** The study of more than two variables at a time is known as multiple regression. Under this, only one variable is taken as a dependent variable and all the other variables are taken as independent variables. For example, If we consider sales as Variable X, advertising as variable Y and Income as Variable Z, then using the functional relation X = *f*(Y, Z), we can find the value of variable X (Sales) if the value of Variable Y (Advertising) and the value of variable Z (Income) is given.

2. Total and Partial Regression

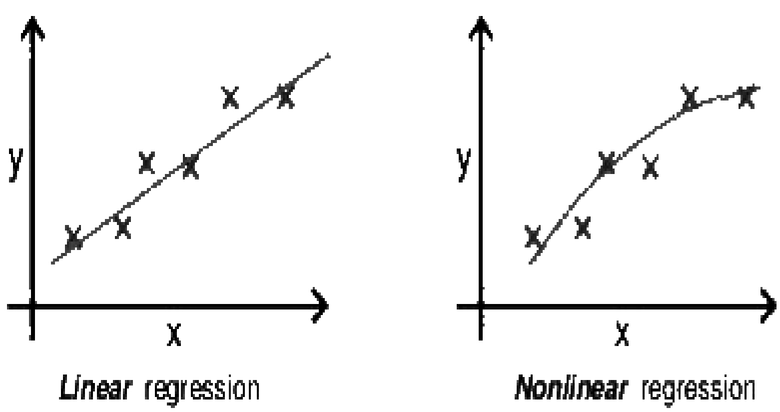
a. Total Regression: Total regression analysis is one in which we study the effect of all the variables simultaneously. For example, when we want to study the effect of advertising expenditure of a business represented by variable Y, income of the consumer represented by variable Z, on the amount of sales represented by variable X, we can study impact of advertising and income simultaneously on sales. This is a case of total regression analysis. In such cases, the regression equation is represented as follows: **X = *f* (Y, Z),**

a. Partial Regression: In the case of Partial Regression one or two variables are taken into consideration and the others are excluded. For example, when we want to study the effect of the advertising expenditure of a business represented by variable Y, income of the consumer represented by variable Z, on the amount of sales represented by variable X, we will not study the impact of both income and advertising simultaneously, rather we will first study the effect of income on sales keeping advertising constant and then effect of advertising on sales keeping income constant. Partial regression can be written as **X=f (Y not Z).**

3. Linear and Non-Linear Regression

a. Linear Regression: When the functional relationship between X and Y is expressed as the first-degree equations, it is known as linear regression. In other words, when the points plotted on a scatter diagram concentrate around a straight line it is the case of linear regression.

b. Non-linear Regression: On the other hand, if the line of regression (in the scatter diagram) is not a straight line, the regression is termed as curved or non-linear regression. The regression equations of non-linear regression are represented by equations of higher degree. The following diagrams show the linear and non-linear regressions:



8.7 RELATIONSHIP BETWEEN CORRELATION AND REGRESSION

1. Correlation is a quantitative tool that measures the degree of relationship that is present between two variables. It shows the degree and direction of the relation between two variables. Regression helps us to find the value of a dependent variable when the value of an independent variable is given.

2. Correlation between two variables is the same. For example, if we calculate the correlation between sales and advertising or advertising and sales, the value of correlation will remain same. But this is not true for Regression. The regression equation of Advertising on sales will be different from regression equation of Sales on advertising.
3. If there is a positive correlation, the distance between the two lines will be less. That means the two regression lines will be closer to each other- Similarly, if there is a low correlation, the lines will be farther from each other. A positive correlation implies that the lines will be upward-sloping whereas a negative correlation implies that the regression lines will be downward sloping.
4. Correlation between two variables can be calculated by taking the square root of the product of the two regression coefficients.

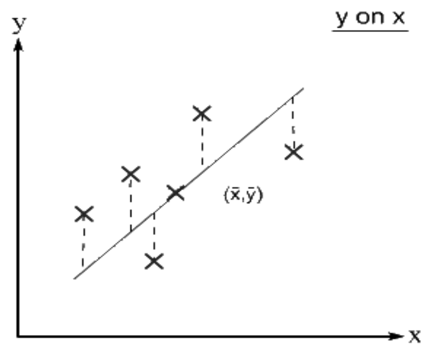
Following are some of the differences between Correlation and Regression:

No.	Correlation	Regression
1.	Correlation measures the degree and direction of a relationship between two variables.	Regression measures the change in the value of a dependent variable given the change in value of an independent variable.
2.	Correlation does not depict a cause-and-effect relationship.	Regression depicts the causal relationship between two variables.
3.	Correlation is a relative measure of the linear relationship that exists between two variables.	Regression is an absolute measure that measures the change in value of a variable.
4.	Correlation between two variables is the same. In other words, Correlation between two variables is the same. rxy =ryx.	Regression is not symmetrical in formation. So, the regression coefficients of X on Y and of Y on X are different.
5.	Correlation is independent of Change in origin or scale.	Regression is independent of Change in origin but not of scale.
6.	Correlation is not capable of any further mathematical treatment.	Regression can be further treated mathematically.
7.	Coefficient of correlation always lies between -1 and +1.	Regression coefficient can have any value.

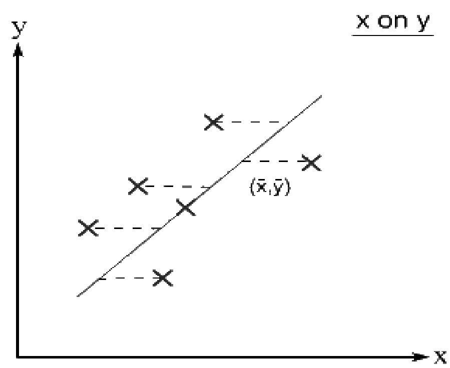
8.8 REGRESSION LINES

The lines that are used in Regression for estimation are called as regression line. In other words, the lines that are used to study the dependence of one variable on the other variable are called as regression line. If we have two variables X and Y then there.

a. Regression Line of Y on X: Regression Line Y on X measures the dependence of Y on X and we can estimate the value of Y for the given values of X. In this line Y is dependent variable and X is independent variable.

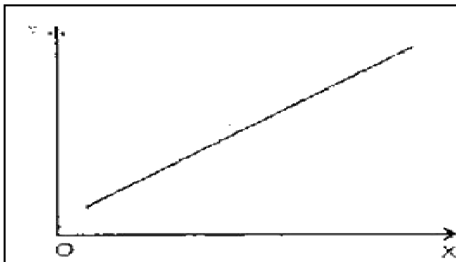


b. Regression Line of X on Y: Regression Line X on Y measures the dependence of X on Y and we can estimate the value of X for the given values of Y. In this line X is dependent variable and Y is independent variable.



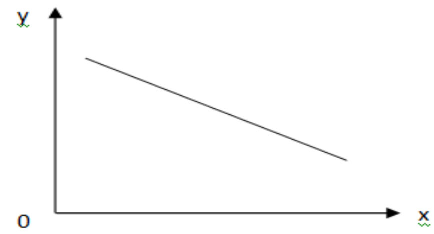
The direction of two regression equation depends upon the degree of correlation between two variables. Following can be the cases of correlation between two variables:

1. Perfect positive correlation: If there is a perfect positive correlation between two variables (i.e. r = +1), both the lines will coincide with each other and will have having positive slope. Both the lines X on Y and Yon X will be the same in this case. In other words, in that case, only one regression line can be drawn as shown in the diagram. The slope of the line will be upward.



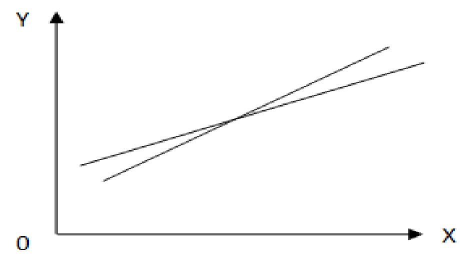
Both the line will coincide with positive slope

2. Perfect negative correlation: If there is a perfect negative correlation between two variables (i.e. r = -1), both the lines will coincide with each other and will in such case these lines will be having a negative slope. Both the lines X on Y and Y on X will be the same but downward sloping. In other words, in that case, only one regression line can be drawn as shown in the diagram. The slope of the line will be upward.



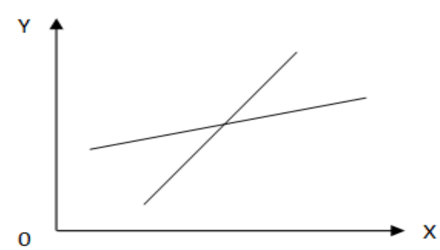
Both the line will coincide with negative slope

3. High degree of correlation: If there is a high degree of correlation between two variables, both the lines will be near to each other. In other words, these lines will be closer to each other but the lines will not coincide with each other. Both the lines will be separate. Further the direction of lines depends upon the positive or negative correlation.



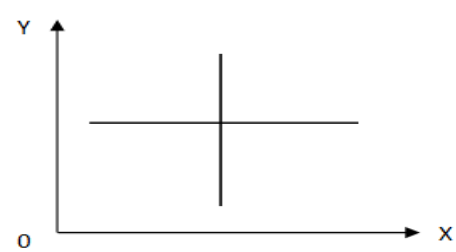
Both the line will be close to each other

4. Low degree of correlation: If there is a low degree of correlation between two variable, both the lines will be have more distance from each other. In other words, these lines will be farther to each other, that is the gap between the two lines will be more. Both the lines will be separate. Further, the direction of lines depends upon the positive or negative correlation.



Both the line will be farther from each other

5. No correlation: If there is no correlation between two variables (i.e. r = 0), both the lines will be perpendicular to each other. In other words, these lines will cut each other at $90^{0.}$ This diagram depicts the perpendicular relation between the two regression lines when there is absolutely zero



Both the line will be perpendicular to each other correlation between the two variables under the study.

8.9 LEAST SQUARE METHOD OF FITTING REGRESSION LINES

Under this method, the lines of best fit are drawn as the lines of regression. These lines of regression are known as the lines of the best fit because, with the help of these lines we can estimate the values of one variable depending on the value of other variables. According to the Least Square method, the regression line should be plotted in such a way that sum of square of the difference between actual value and an estimated value of the dependent variable should be least or minimum possible. Under this method we draw two regression lines that are

a. **Regression line Y on X**: it measures the value of Y when value of X is given. In other words, it assumes that X is an independent variable whereas the other variable Y is a dependent variable. Mathematically this line is represented by

$$Y = a + bX$$

Where Y – Dependent Variable

X – Independent Variable

a & b – Constants

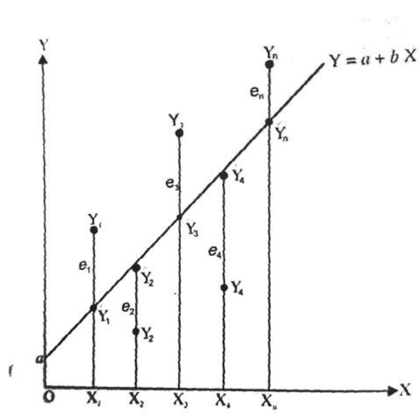
b. **Regression line X on Y** – it measures the value of X when value of Y is given. In other words, it assumes that Y is an independent variable whereas the other variable X is a dependent variable. Mathematically this line is represented by

$$X = a + bY$$

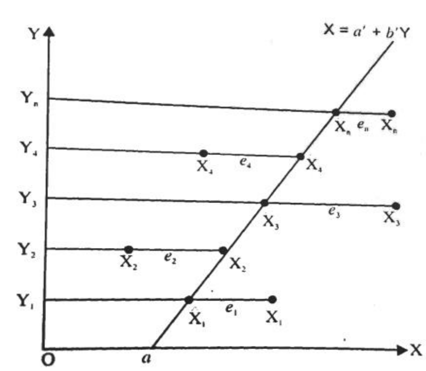
Where X – Dependent Variable

Y – Independent Variable

a & b – Constants



Equation Y on X



Equation X on Y

In the above two regression lines, there are two constants represented by “a” and “b”. The constant “b” is also known as regression coefficient, which is denoted as “byx” and “bxy”, Where “byx” represent regression coefficient of equation Y on X and “bxy” represent regression coefficient of equation X on Y. When the value of these two variables “a” and “b” is determined we can find out the regression line.

8.10 DIRECT METHODS TO ESTIMATE REGRESSION EQUATION

The regression equations can be obtained by 'Normal Equation Method" as follows:

1. **Regression Equation of Y on X:** The regression equation Y on X is in the format of Y = a + bx, where Y is a Dependent Variable and X is an Independent Variable. To estimate this regression equation, the following normal equations are used:

   $$\Sigma Y = na + b_{yx}\Sigma X$$
   $$\Sigma XY = a\,\Sigma X + b_{yx}\Sigma X^2$$

   With the help of these two equations the values of ‘a' and 'b' are obtained and by putting the values of ‘a' and 'b' in the equation Y = a + bX we can predict or estimate value of Y for any value of X.

2. **Regression Equation of X on Y:** The regression equation X on Y is in the format of X = a + bY, where X is a Dependent Variable and Y is an Independent Variable. To estimate this regression equation, following normal equations are used:

   $$\Sigma X = na + b_{xy}\Sigma Y$$
   $$\Sigma XY = a\,\Sigma Y + b_{xy}\Sigma Y^2$$

   With the help of these two equations the values of ‘a' and 'b' are obtained and by putting the values of ‘a' and 'b' in the equation X = a + bY we can predict or estimate value of Y for any value of X.

Example 1. Find out the two regression lines for the data given below using the method of least square.

Variable X:	5	10	15	20	25
Variable Y:	20	40	30	60	50

Determination of the regression lines by the method of least square. Also find out

a. Value of Y when value of X is 40
b. Value of X when value of Y is 80.

Solution:

X	**Y**	**X2**	**Y2**	**XY**

5	20	25	400	100
10	40	100	1600	400
15	30	225	900	450
20	60	400	3600	1200
25	50	625	2500	1250
XX = 75	XY = 200	XX2=1375	XY2=9,000	XXY =3400

(i) Regression line of Y on X

This is given by Y = *a + bX*

where *a* and *b* are the two constants which are found by solving simultaneously the two normal equations as follows:

$\Sigma Y = na + b_{yx}\Sigma X$

$\Sigma XY = a \Sigma X + b_{yx}\Sigma X^2$

Substituting the given values in the above equations we get,

200 = 5a + 75b ……………………………………. (i)

3400 =75a + 1375b ……………………………………. (ii)

Multiplying the eqn. (i) by 15 we get

3000 = 75a + 1125b……………………………………. (iii)

Subtracting the equation (iii) from equation (ii) we get,

3400 =75a + 1375b

-3000 = -75a - 1125b

400 = 250b

or b = 1.6

Putting the above value of b in the eqn. (i) we get,

200 = 5a+ 75(1.6) or

5a =200- 120 or

a = 16

Thus, a = 16, and b = 1.6

Putting these values in the equation Y = *a* + bX we get

Y = 16+ 1.6X

So, when X is 40, the value of Y will be

Y = 16+ 1.6(40) = 80

(ii) Regression line of X on Y

This is given by X = *a + bY*

where *a* and *b* are the two constants which are found by solving simultaneously the two normal equations as follows:

$\Sigma X = na + b_{xy} \Sigma Y$

$\Sigma XY = a \Sigma Y + b_{xy} \Sigma Y^2$

Substituting the given values in the above equations we get,

75 = 5a+200b ……….…………………………………….. (i)

3400 = 200a + 9000b …………………………………….. (ii)

Multiplying the eqn. (i) by 40 we get

3000 = 200a + 8000b…………………………………….. (iii)

Subtracting the equation (iii) from equation (ii) we get,

3400 = 200a + 9000b

<u>-3000 = -200a + -8000b</u>

400 = 1000b

or b = .4

Putting the above value of b in the eqn. (i) we get,

75 = 5a + 200(.4) or

5a = -5 or

a = -1

Thus, a = -1, and b = .4

Putting these values in the equation X = $a + bY$ we get

X = -1+ .4Y

So, when Y is 80, the value of X will be

X = -1+ .4(80) = 31

8.11 OTHER METHODS OF ESTIMATING REGRESSION EQUATION

This method discussed above is known as the direct method. This is one of the popular methods of finding the regression equation. But sometimes this method of finding regression equations becomes cumbersome and lengthy especially when the values of X and Y are very large. In this case, we can simplify the calculation by taking the deviations of X and Y than dealing with actual values of X and Y. In such case

Regression equation Y on X

Y = a + bX

will be converted to $(Y - \bar{Y}) = byx\,(X - \bar{X})$

Similarly, Regression equation X on Y:

$X = a + bY$

will be converted into $(X - \bar{X}) = bxy\,(Y - \bar{Y})$

Now when we are using these regression equations, the calculations will become very simple as now we have to calculate value of only one constant that is the value of "b" which is our regression coefficient. As there are two regression equations, so we need to calculate two regression coefficients that is Regression Coefficient X on Y, which is symbolically denoted as "bxy" and similarly Regression Coefficient Y on X, which is denoted as "byx". However, these coefficients can also be calculated using different methods. As we take deviations under this method, we can take deviations using actual mean, assumed mean or we can calculate it by not taking the deviations. Following formulas are used in such cases:

Method	Regression Coefficient X on Y	Regression Coefficient Y on X
When deviations are taken from actual mean	$bxy = \frac{\sum xy}{\sum y^2}$	$byx = \frac{\sum xy}{\sum x^2}$
When deviations are taken from assumed mean	$bxy = \frac{N\sum dxdy - \sum dx \sum dy}{N\sum dy^2 - (\sum dy)^2}$	$byx = \frac{N\sum dxdy - \sum dx \sum dy}{N\sum dx^2 - (\sum dx)^2}$
Direct Method: Using sum of X and Y	$bxy = \frac{N\sum XY - \sum X \sum Y}{N\sum Y^2 - (\sum Y)^2}$	$byx = \frac{N\sum XY - \sum X \sum Y}{N\sum X^2 - (\sum X)^2}$
Using the correlation coefficient (r) and standard deviation (σ)	$bxy = r \cdot \frac{\sigma x}{\sigma y}$	$byx = r \cdot \frac{\sigma y}{\sigma x}$

Example 2. From the information give below obtain two regression lines X on Y and Yon X using

1. Actual Mean Method
2. Assumed Mean Method
3. Direct Method (Without taking Mean)

Number of hours machine-operated	7	8	6	9	11	9	10	12
Production (Units in 000):	4	5	2	6	9	5	7	10

Solution: 1. <u>Actual Mean Method</u>

Calculation of Regression Equation

X	Y	x =X - $\bar{X}$	x^2	y =Y - $\bar{Y}$	y^2	xy
7	4	-2	4	-2	4	4
8	5	-1	1	-1	1	1
6	2	-3	9	-4	16	12
9	6	0	0	0	0	0
11	9	2	4	3	9	6
9	5	0	0	-1	1	0
10	7	1	1	1	1	1
12	10	3	9	4	16	12
∑X =72	**∑Y =48**		**$\sum x^2$ = 28**		**$\sum y^2$ = 48**	**∑xy = 36**

$\bar{X} = \frac{\sum X}{N} = \frac{72}{8} = 9$, $\bar{Y} = \frac{\sum Y}{N} = \frac{48}{8} = 6$

Regression equation of X on Y:

$(X - \bar{X}) = b_{xy} (Y - \bar{Y})$

Where bxy = $\frac{\sum xy}{y^2}$

$= \frac{36}{48} = .75$

So (X – 9) = .75 (Y – 6)

X – 9 = .75Y – 4.5

X = 4.5 + .75Y

Regression equation of Y on X:

$(Y - \bar{Y}) = b_{xy} (X - \bar{X})$

Where byx = $\frac{\sum xy}{x^2}$

$= \frac{36}{28} = 1.286$

So (Y – 6) = 1.286 (X – 9)

Y – 6 = 1.286X – 11.57

Y = – 5.57+ 1.286X

1. **<u>Assumed Mean Method</u>**

Calculation of Regression Equation

X	Y	dx =X – A (A = 8)	dx^2	dy =Y – A (A = 5)	dy^2	dxdy
7	4	-1	1	-1	1	1
8	5	0	0	0	0	0
6	2	-2	4	-3	9	6
9	6	1	1	1	1	1

11	9	3	9	4	16	12
9	5	1	1	0	0	0
10	7	2	4	2	4	4
12	10	4	16	5	25	20
$\sum X$ =72	**$\sum Y$ =48**	**$\sum dx$ = 8**	**$\sum dx^2$ = 36**	**$\sum dy$ = 8**	**$\sum dy^2$ = 56**	**$\sum xy$ = 44**

$\bar{X} = \frac{\sum X}{N} = \frac{72}{8} = 9$, $\bar{Y} = \frac{\sum Y}{N} = \frac{48}{8} = 6$

Regression equation of X on Y:

$(X - \bar{X}) = b_{xy} (Y - \bar{Y})$

Where $bxy = \frac{N\sum dxdy - \sum dx \sum dy}{N\sum dy^2 - (\sum dy)^2}$

$= \frac{8\ (44) - (8)\ (8)}{8\ (56) - (8)^2} = \frac{352 - 64}{448 - 64} = \frac{288}{384} = .75$

So (X – 9) = .75 (Y – 6)

X – 9 = .75Y – 4.5

X = 4.5 + .75Y

Regression equation of Y on X:

$(Y - \bar{Y}) = b_{xy} (X - \bar{X})$

Where $byx = \frac{N\sum dxdy - \sum dx \sum dy}{N\sum dx^2 - (\sum dx)^2}$

$= \frac{8\ (44) - (8)\ (8)}{8\ (36) - (8)^2} = \frac{288}{224} = 1.286$

So (Y – 6) = 1.286 (X – 9)

Y – 6 = 1.286X – 11.57

Y = – 5.57+ 1.286X

2. <u>Direct Method (Without taking Mean)</u>

Calculation of Regression Equation

X	Y	X^2	Y^2	XY
7	4	49	16	28
8	5	64	25	40
6	2	36	4	12
9	6	81	36	54
11	9	121	81	99
9	5	81	25	45
10	7	100	49	70
12	10	144	100	120
$\sum X$ =72	**$\sum Y$ =48**	**$\sum X^2$ = 676**	**$\sum Y^2$ = 336**	**$\sum XY$ = 468**

$\bar{X} = \frac{\sum X}{N} = \frac{72}{8} = 9,$ $\bar{Y} = \frac{\sum Y}{N} = \frac{48}{8} = 6$

Regression equation of X on Y:

$(X - \bar{X}) = b_{xy} (Y - \bar{Y})$

Where $bxy = \frac{N\sum XY - \sum X \sum Y}{N\sum Y^2 - (\sum Y)^2}$

$= \frac{8\ (468) - (72)\ (48)}{8\ (336) - (48)^2}$

$= \frac{3744 - 3456}{2688 - 2304} = \frac{288}{384} = .75$

So $(X - 9) = .75\ (Y - 6)$

$X - 9 = .75Y - 4.5$

X = 4.5 + .75Y

Regression equation of Y on X:

$(Y - \bar{Y}) = b_{xy} (X - \bar{X})$

Where $byx = \frac{N\sum XY - \sum X \sum Y}{N\sum X^2 - (\sum X)^2}$

$= \frac{8\ (468) - (72)\ (48)}{8\ (676) - (72)^2}$

$= \frac{3744 - 3456}{5408 - 5184} = \frac{288}{224} = 1.286$

So $(Y - 6) = 1.286\ (X - 9)$

$Y - 6 = 1.286X - 11.57$

Y = – 5.57+ 1.286X

Example 3. Find out two Regression equations on basis of the data given below:

	X	Y
Mean	60	80
Standard Deviation (S.D.)	16	20
Coefficient of Correlation	.9	

Also find value of X when Y = 150 and value of Y when X = 100.

Solution: Regression equation of X on Y:

$(X - \bar{X}) = b_{xy} (Y - \bar{Y})$

Where $bxy = r \frac{\sigma x}{\sigma y}$

$= .9\ \frac{16}{20} = .72$

So $(X - 60) = .72 (Y - 80)$

$X - 60 = .72Y - 57.6$

$\mathbf{X = 2.4 + .72Y}$

When Y = 150 than X = 2.4 + .72(150) = 110.4

Regression equation of Y on X:

$(Y - \bar{Y}) = b_{xy} (X - \bar{X})$

Where $byx = r \frac{\sigma y}{\sigma x}$

$= .9 \frac{20}{16} \quad = 1.125$

So $(Y - 80) = 1.125 (X - 60)$

$Y - 80 = 1.125X - 67.5$

$\mathbf{Y = 12.5 + 1.125 X}$

When X = 100 than Y = 12.5 + 1.125 (100) = 125

8.12 PROPERTIES OF REGRESSION COEFFICIENTS

The regression coefficients discussed above have a number of properties which are given as under:

1. The geometric Mean of the two regression coefficients gives the coefficients of correlation i.e.
$r = \sqrt{bxy * byx}$

2. Both the regression coefficients must have the same sign i.e. in other words either both coefficients will have + signs or both coefficients will have - signs. This is due to the fact that in first property we have studied that geometric means of both coefficients will give us value of correlation. If one sign will be positive and other will be negative, the product of both signs will be negative. And it is not possible to find out correlation of negative value.

3. The signs of regression coefficients will give us signs of coefficient of correlation. This means if the regression coefficients are positive the correlation coefficient will be positive, and if the regression coefficients are negative then the correlation coefficient will be negative.

4. If one of the regression coefficients is greater than unity or 1, the other must be less than unity. This is because the value of coefficient of correlation must be in between ± 1. If both the regression coefficients are more than 1, then their geometric mean will be more than 1 but the value of correlation cannot exceed 1.

5. The arithmetic mean of the regression coefficients is either equal to or more than the correlation coefficient $\frac{bxy+byx}{2} \geq \sqrt{bxy * byx}$

6. If the regression coefficients are given, we can calculate the value of standard deviation by using the following formula.

a. $bxy = r\,\frac{\sigma x}{\sigma y}$ or $byx = r\,\frac{\sigma y}{\sigma x}$

7. Regression coefficients are independent of change of origin but not of scale. This means that if the original values of the two variables are added or subtracted by some constant, the values of the regression coefficients will remain the same. But if the original values of the two variables are multiplied, or divided by some constant (common factors) the values of the regression equation will not remain the same.

Example 4. From the following data find out two lines of regression land also find out value of correlation.

$\sum X = 250$; $\sum Y = 300$; $\sum XY = 7900$; $\sum X^2 = 6500$; $\sum Y^2 = 10000$; n = 10

Solution:

$\overline{X} = \frac{\sum X}{N} = \frac{250}{10} = 25$, $\overline{Y} = \frac{\sum Y}{N} = \frac{300}{10} = 30$

Regression equation of Y on X:

$(Y - \overline{Y}) = b_{xy}\,(X - \overline{X})$

Where $byx = \frac{N\sum XY - \sum X\sum Y}{N\sum X^2 - (\sum X)^2}$

$= \frac{10\,(7900) - (250)\,(300)}{10\,(6500) - (250)^2}$

$= \frac{79000 - 75000}{65000 - 62500} = \frac{4000}{2500} = 1.6$

So $(Y - 30) = 1.6\,(X - 25)$

$Y - 30 = 1.6X - 40$

Y = – 10+ 1. 6 X

Regression equation of X on Y:

$(X - \overline{X}) = b_{xy}\,(Y - \overline{Y})$

Where $bxy = \frac{N\sum XY - \sum X\sum Y}{N\sum Y^2 - (\sum Y)^2}$

$= \frac{10\,(7900) - (250)\,(300)}{10\,(10000) - (300)^2} = \frac{79000 - 75000}{100000 - 90000} = \frac{4000}{10000} = .4$

So $(X - 25) = .4\,(Y - 30)$

$X - 25 = .4Y - 12$

X = 13 + .4Y

Coefficients of Correlation

r = $\sqrt{bxy \,*\, byx}$

r = $\sqrt{1.6 \,*\, 0.4}$

r = $\sqrt{.64}$

r = .8

Example 5. From the following data find out two lines of regression land also find out value of correlation. Also find value of Y when X = 30

$\sum X = 140$; $\sum Y = 150$; $\sum(X - 10)(Y - 15) = 6$;

$\sum (X - 10)^2 = 180$; $\sum (Y - 15)^2 = 215$; n = 10

Solution:

Let's take assumed mean of Series X = 10 and Series Y = 15.

$\sum dx = \sum (X - 10) = \sum X - 10n = 140 - 100 = 40$

$\sum dy = \sum (Y - 15) = \sum Y - 15n = 150 - 150 = 0$

$\sum dx^2 = \sum (X - 10)^2 = 180$

$\sum dy^2 = \sum (Y - 15)^2 = 215$

$\sum dxdy = \sum (X - 10)(Y - 15) = 6$

So,

$\overline{X} = \mathbf{A} + \frac{\sum X}{N} = 10 + \frac{40}{10} = 14$

$\overline{Y} = A + \frac{\sum Y}{N} = 15 + \frac{0}{10} = 15$

Regression equation of Y on X:

$(Y - \overline{Y}) = b_{xy} (X - \overline{X})$

Where byx = $\frac{N\sum dxdy - \sum dx \sum dy}{N\sum dx^2 - (\sum dx)^2}$

$= \frac{10(6) - (40)(0)}{10(180) - (40)^2} = \frac{60}{200} = .3$

So (Y – 15) = .3 (X – 14)

Y – 15 = .3X – 4.2

Y = 10.8+ .3X

When X = 30 than Y = 10.8 + .3(30) = 19.8

Regression equation of Y on X:

$(Y - \overline{Y}) = b_{xy} (X - \overline{X})$

Where $byx = \frac{N\sum dxdy - \sum dx \sum dy}{N\sum dx^2 - (\sum dx)^2}$

$= \frac{10\ (6) - (40)\ (0)}{10\ (25) - (0)^2} \quad = \quad \frac{60}{250} = .24$

So (Y – 15) = .24 (X – 14)

Y – 15 = .24X – 3.36

Y = 11.64+ .24X

Coefficients of Correlation

$r = \sqrt{bxy\ *\ byx} = \sqrt{.3\ *\ .24}$

$r = \sqrt{.072}$

r = .268

Example 5. From the following data find out which equation is equation X on Y and which equation is equation Y on X. Also find $\overline{X}$, $\overline{Y}$ and r.

3X + 2Y – 26 = 0

6X + Y – 31 = 0

Solution: To find $\overline{X}$ and $\overline{Y}$, we will solve following simultaneous equations

3X + 2Y = 26 …………………………….. (i)

6X + Y = 31 ……………………………. (ii)

Multiply equation (i) with 2, we get

6X + 4Y = 52 …………………………….. (iii)

Deduct equation (ii) from equation (iii)

6X + 4Y = 52

-<u>6X - Y = -31</u>

3Y = 21

Y = 7

Or $\overline{Y}$ = 7.

Put the value of Y in Equation (i), we get

3X + 2(7) = 26

3X + 14 = 26

3X = 12

X = 4

or $\overline{X}$ = 4

Let 3X + 2Y = 26 be regression equation X on Y

3X = 26 – 2Y

X = $\frac{26}{3} - \frac{2}{3}$ Y

So $b_{xy} = -\frac{2}{3}$

Let 6X + Y = 31 be regression equation Y on X

Y = 31 – 6X

So b_{yx} = – 6

As r = $\sqrt{bxy \ * \ byx}$

r = $-\sqrt{-(\frac{2}{3}) \ \times (-.6 \ \ )}$

r = -2, but this is not possible as value of r always lies between $-1 \ and + 1$. So, our assumption is wrong and equation are reverse.

Let 6 X + Y = 31 be regression equation X on Y

6X = 31 –Y

X = $\frac{31}{6} - \frac{1}{6}$ Y

So $b_{xy} = -\frac{1}{6}$

Let 3X + 2Y = 26 be regression equation Y on X

2Y = 26 – 3X

Y = $\frac{26}{2} - \frac{3}{2}$ X

So $b_{yx} = -\frac{3}{2}$

As r = $\sqrt{bxy \ * \ byx}$

r = $-\sqrt{-(\frac{1}{6}) \ \times -(\frac{3}{2}) \ \ }$

r = $-.5$, which is possible. So, our assumption is right.

So, **$\overline{Y}$ = 7;**

$\overline{X}$ = 4;

X on Y is X = $\frac{31}{6} - \frac{1}{6}$ Y

Y on X is Y = $\frac{26}{2} - \frac{3}{2}$ **X**

r = $-.5$

TEST YOUR UNDERSTANDING (D)

1. Find both regression equations:

X	6	2	10	4	8
Y	9	11	5	8	7

2. From following estimate the value of Y when X = 30 using regression equation.

X	25	22	28	26	35	20	22	40	20	18	19	25
Y	18	15	20	17	22	14	15	21	15	14	16	17

3. Fit two regression lines:

X	30	32	38	35	40
Y	10	14	16	20	15

Find X when Y = 25 and find Y when X = 36.

4. Find out two Regression equations on basis of the data given below:

	X	Y
Mean	65	67
Standard Deviation (S.D.)	2.5	3.5
Coefficient of Correlation	.8	

5. In a data the Mean values of X and Y are 20 and 45 respectively. Regression coefficient b_{yx} = 4 and b_{xy} 1/9. Find

a. coefficient of correlation
b. Standard Deviation of X, if S.D. of Y = 12
c. Find two regression lines

6. You are supplied with the following information. Variance of X = 36

12X – 51Y + 99 = 0

60X – 27Y = 321.

Calculate (a) The average values of X and Y

(b) The standard deviation of Y and

7. The lines of regression of Y on X and X on Y are Y = X + 5 and 16X = 9Y + 4 respectively. Also, σy=4, Find $\bar{X}$, $\bar{Y}$, σx and r.

8. Given:

$$\sum X = 56, \sum Y = 40, \sum X^2 = 524$$

$$\sum Y^2 = 256, \sum XY = 364, N = 8$$

(a) $find\ the\ regression\ equatiopn\ of\ X\ on\ Y$

Answers

1) X = 16.4 – 1.3Y, Y = 11.9 - .65X   2) 18.875

3) Y = .46X – 1.1, X = .6Y + 26, Value of Y = 15.46, Value of X = 40.25

4) Y = 1.12X – 5.8, X = .57Y +26,81,   5) 0.67, 2, Y= 4X – 35 and X = 1/9 Y +15

6) Mean of X = 13, Mean of Y= 17, S.D of Y = 8

7) Mean of X = 7, Mean of Y= 12, S.D of X = 3, r= .75,   8) X = 1.5Y - 0.5, r = .977

8.13 SUM UP

- Regression is a useful tool of forecasting.
- With help of regression, we can predict the value of can find the value of X if value of Y is given or value of Y if value of X is given.
- It creates the mathematical linear relation between two variables X and Y, out of which one variable is dependent and other is independent.
- In this we find out two regression equations.
- Regression can be linear or nonlinear.
- It can be simple or multiple.
- Regression is based on the principle of Least Squares.
- We can also find out correlation coefficient with help of regression coefficients.

8.14 KEY TERMS

- **Regression:** Regression creates the mathematical linear relation between two variables X and Y, out of which one variable is dependent and other is independent.
- **Simple Regression**: When there are only two variables under study it is known as a simple regression. For example, we are studying the relation between Sales and Advertising expenditure.

- **Multiple Regression:** The study of more than two variables at a time is known as multiple regression. Under this, only one variable is taken as a dependent variable and all the other variables are taken as independent variables.
- **Total Regression:** Total regression analysis is one in which we study the effect of all the variables simultaneously.
- **Partial Regression:** In the case of Partial Regression one or two variables are taken into consideration and the others are excluded.
- **Linear Regression:** When the functional relationship between X and Y is expressed as the first-degree equations, it is known as linear regression. In other words, when the points plotted on a scatter diagram concentrate around a straight line it is the case of linear regression.
- **Non-linear Regression**: On the other hand, if the line of regression (in scatter diagram) is not a straight line, the regression is termed as curved or non-linear regression.
- **Least Square method**: According to the Least Square method, regression line should be plotted in such a way that sum of square of the difference between actual value and estimated value of the dependent variable should be least or minimum possible.

8.15 QUESTIONS FOR PRACTICE

Q1. What is Regression? What are uses of Regression.

Q2. What is relation between Regression and correlation?

Q3. Explain different types of regressions.

Q4. How two regression lines are determined under direct method?

Q5. Explain various methods of finding regression equations.

Q6. What are limitations of regression analysis?

Q7. What are properties of regression coefficients?

8.16 FURTHER READINGS

- J. K. Sharma, *Business Statistics,* Pearson Education.
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